

Dependence of Discrete Sturm-Liouville Eigenvalues on Problems

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*Dedicated to the memory of the fourth author, Professor Hongyou Wu
(1962.11–2009.6.9).*

Abstract. This paper is concerned with dependence of discrete Sturm-Liouville eigenvalues on problems. Topologies and geometric structures on various spaces of such problems are firstly introduced. Then, relationships between the analytic and geometric multiplicities of an eigenvalue are discussed. It is shown that all problems sufficiently close to a given problem have eigenvalues near each eigenvalue of the given problem. So, all the simple eigenvalues live in so-called continuous simple eigenvalue branches over the space of problems, and all the eigenvalues live in continuous eigenvalue branches over the space of self-adjoint problems. The analyticity, differentiability and monotonicity of continuous eigenvalue branches are further studied.

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1. Introduction

A discrete Sturm-Liouville problem (briefly, SLP) considered in the present paper consists of a discrete Sturm-Liouville equation (briefly, SLE)

$$-\nabla(f_n \Delta y_n) + q_n y_n = \lambda w_n y_n, \quad n \in [1, N], \quad (1.1)$$

and the boundary condition (briefly, BC)

$$A \begin{pmatrix} y_0 \\ f_0 \Delta y_0 \end{pmatrix} + B \begin{pmatrix} y_N \\ f_N \Delta y_N \end{pmatrix} = 0, \quad (1.2)$$

where $N \geq 2$ is an integer, Δ and ∇ are the forward and backward difference operators, respectively, i.e., $\Delta y_n = y_{n+1} - y_n$ and $\nabla y_n = y_n - y_{n-1}$; $f =$

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$\{f_n\}_{n=0}^N$, $q = \{q_n\}_{n=1}^N$ and $w = \{w_n\}_{n=1}^N$ are complex-valued sequences such that

$$f_n \neq 0 \text{ for } n \in [0, N], \quad w_n \neq 0 \text{ for } n \in [1, N]; \quad (1.3)$$

λ is the spectral parameter; the interval $[M, N]$ denotes the set of integers $\{M, M+1, \dots, N\}$; and A and B are 2×2 complex matrices such that

$$\text{rank}(A, B) = 2. \quad (1.4)$$

Throughout this paper, by \mathbb{C} , \mathbb{R} , and \mathbb{Z} denote the sets of the complex numbers, real numbers, and integer numbers, respectively; and by \bar{z} denotes the complex conjugate of $z \in \mathbb{C}$. Moreover, when a capital Latin letter stands for a matrix, the entries of the matrix are denoted by the corresponding lower case letter with two indices. For example, the entries of a matrix C are c_{ij} 's.

The dependence of the continuous Sturm-Liouville eigenvalues on the problems and its applications have been extensively studied (cf., [3, 4, 5, 9, 10, 13, 14, 15, 16, 18, 19, 27]). In [16], Kong and Zettl proved that the eigenvalues of continuous SLPs depend not only continuously but also smoothly on problems and then gave an expression for the derivative of the n -th eigenvalue with respect to a given parameter in the continuous SLP. Later, they, together with Wu, gave a natural geometric structure on the space of BCs of continuous SLPs in [15]. This structure is the base for studying the dependence of Sturm-Liouville eigenvalues on the BCs. In addition, they investigated the differentiability of continuous eigenvalue branches based on this structure, and discussed the relationships between the algebraic and geometric multiplicities of an eigenvalue.

Along another line, research on discrete spectral problems and their inverse problems has been of growing interest in recent years (cf., e.g. [1, 2, 6, 7, 8, 12, 17, 21, 22, 23, 24, 25, 26]). Atkinson [1] and Jirari [12] studied spectral problems of second-order discrete scalar self-adjoint SLPs with separate BCs. In [23], the third author of the present paper with her coauthor Chen investigated the following vector difference equation

$$-\nabla(C_n \Delta y_n) + B_n y_n = \lambda w_n y_n, \quad n \in [1, N], \quad N \geq 2, \quad (1.5)$$

with the general boundary condition

$$R \begin{pmatrix} -y_0 \\ y_N \end{pmatrix} + S \begin{pmatrix} C_0 \Delta y_0 \\ C_N \Delta y_N \end{pmatrix} = 0, \quad (1.6)$$

where C_n ($n \in [0, N]$), B_n , and w_n ($n \in [1, N]$) are Hermitian $d \times d$ matrices, C_0 and C_N are nonsingular, $w_n > 0$ for $n \in [1, N]$; R and S are $2d \times 2d$ matrices with $\text{rank}(R, S) = 2d$. It is evident that the BC (1.2) is included in the BC (1.6). The spectral results obtained in [23] will be used in the study of the multiplicity of eigenvalues in the present paper. Further, the third author of the present paper with her coauthor Lv studied error estimate of eigenvalues of perturbed problems, sufficiently close to a given Sturm-Liouville problem (1.5) and (1.6), by some variational properties of the eigenvalues under a

certain non-singularity condition in [17]. So we obtained the continuous dependence of eigenvalues on problems under the nonsingularity condition.

In Chapter 2 of [13], Kato investigated perturbation problems for linear operators in finite-dimensional spaces. He studied how the eigenvalues change with the operator, in particular when the operator analytically depends on a parameter. His method is based on function-theoretic study of the corresponding resolvent. Obviously, the eigenvalue problem of the self-adjoint discrete SLPs consisting of (1.1)-(1.2) corresponds to that of an operator in a finite-dimensional space. Note that the operator defined by (1.1)-(1.2) may be multi-valued since $x(0)$ and $x(N+1)$ may not be uniquely determined by the BC (1.2), and the problem discussed in the present paper is dependent on multi-parameters. However, the operators are all single-valued and their perturbations are only referred to one single parameter in [13]. So the results in [13] can not be directly available in our study.

In the present paper, we shall investigate dependence of eigenvalues on the SLP consisting of (1.1) and (1.2). There are two main motivations for our study. Firstly, it is helpful to clarify the common features and differences between the class of continuous SLPs and that of discrete SLPs. Secondly, it is hoped that findings of such work will form a theoretical foundation for numerical works on discrete SLPs and their inverse problems, and such numerical works will shed light on numerical works on continuous SLPs and their inverse problems. Many results in the continuous case may be obtained from the corresponding results in the discrete case, via certain limit procedures, but not vice versa; while some results in the discrete case have relatively direct proofs. In this way, shorter proofs of results in the continuous case may be found.

This paper is organized as follows. In Section 2, we give topologies and geometric structures on various spaces of discrete SLPs, which are fundamental for further developments. In Section 3, we first discuss properties of the analytic and geometric multiplicities of eigenvalues of the discrete SLPs and their relationships, and then study continuous dependence of eigenvalues on the problems. In Section 4, we investigate some fundamental properties of continuous eigenvalue branches including their analyticity, differentiability and monotonicity. Finally, several examples illustrating results of these sections are presented in Section 5.

Remark 1.1. We shall apply the results obtained in the present paper to study some other topics about discrete Sturm-Liouville problems, including dependence of the n -th eigenvalue on problems, inequalities among eigenvalues for different problems, and index problems for eigenvalues in our forthcoming papers.

2. Spaces of problems

In this section, we shall first introduce the topologies and geometric structures on the spaces of discrete SLEs, BCs, and self-adjoint BCs, separately, and

then give the geometric structures of the spaces of discrete SLPs and self-adjoint discrete SLPs. On the one hand, unlike in the continuous case, the space of discrete SLEs in this paper has an easy and obvious structure. On the other hand, the space of BCs and the space of self-adjoint BCs have the same geometric structures as those in the continuous case.

Let the discrete SLE (1.1) be abbreviated as $(1/f, q, w)$. Then the space of discrete SLEs can be written as

$$\Omega_N^{\mathbb{C}} := \{(1/f, q, w) : (1.3) \text{ holds}\}$$

and is equipped with the topology deduced from the complex space \mathbb{C}^{3N+1} . Bold faced lower case Greek letters, such as ω , are used to denote elements of $\Omega_N^{\mathbb{C}}$. The subspace $\Omega_N^{\mathbb{R}}$ of $\Omega_N^{\mathbb{C}}$ has its obvious meaning. For convenience, the maximum norm on \mathbb{C}^{3N+1} will be used:

$$\|(1/f, q, w)\| = \max \left\{ |1/f_0|, \max_{n \in [1, N]} \{|1/f_n|, |q_n|, |w_n|\} \right\}.$$

Note that $\Omega_N^{\mathbb{C}}$ is a connected open subset of \mathbb{C}^{3N+1} . Similarly, $\Omega_N^{\mathbb{R}}$ is an open subset of \mathbb{R}^{3N+1} and has 2^{2N+1} connected components, two of which are

$$\Omega_N^{-, \mathbb{R}, +} := \{(1/f, q, w) \in \Omega_N^{\mathbb{R}} : f_n < 0 \text{ for } n \in [0, N], w_n > 0 \text{ for } n \in [1, N]\},$$

$$\Omega_N^{+, \mathbb{R}, +} := \{(1/f, q, w) \in \Omega_N^{\mathbb{R}} : f_n > 0 \text{ for } n \in [0, N], w_n > 0 \text{ for } n \in [1, N]\}.$$

We also set

$$\Omega_N^{\mathbb{R}, +} := \{(1/f, q, w) \in \Omega_N^{\mathbb{R}} : w_n > 0 \text{ for } n \in [1, N]\},$$

which has 2^{N+1} connected components.

Since equivalent linear algebraic systems of the form (1.2) define the same BC following [15], we will take the quotient space

$$\mathcal{A}^{\mathbb{C}} := M_{2,4}^*(\mathbb{C}) / GL(2, \mathbb{C}), \quad (2.1)$$

equipped with the quotient topology, as the space of BCs, where

$$M_{2,4}^*(\mathbb{C}) := \{2 \times 4 \text{ complex matrix } (A, B) : (1.4) \text{ holds}\},$$

$$GL(2, \mathbb{C}) := \{2 \times 2 \text{ complex matrix } T : \det T \neq 0\}.$$

Note that $M_{2,4}^*(\mathbb{C})$ is an open subspace of $\mathbb{C}^{2 \times 4}$. (2.1) implies that $(A_1, B_1) \sim (A, B)$ if $(A_1, B_1) = T(A, B)$, $T \in GL(2, \mathbb{C})$. Each BC is an equivalence class of coefficient matrices of systems of the form (1.2); that is, an element of $\mathcal{A}^{\mathbb{C}}$. We use $[A | B]$ to denote the BC represented by the system (1.2). Bold faced capital Latin letters, such as \mathbf{A} , are also used for BCs.

Note that the space of BCs is independent of the equation (1.1) either in the continuous or in the discrete case, and so it has the same topology and geometric structure in the discrete case as that in the continuous case. For convenience, we present them as follows. We refer to Theorems 3.1 and 3.3 in [15] for details.

Theorem 2.1. *The space $\mathcal{A}^{\mathbb{C}}$ of BCs is a connected and compact complex manifold of complex dimension 4, while the space $\mathcal{A}^{\mathbb{R}}$ of real BCs is a connected and compact real-analytic manifold of dimension 4.*

In addition, $\mathcal{A}^{\mathbb{C}}$ has the following canonical atlas of local coordinate systems:

$$\begin{aligned}
\mathcal{N}_{1,2}^{\mathbb{C}} &= \left\{ \begin{bmatrix} 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{bmatrix} : b_{ij} \in \mathbb{C}, i, j = 1, 2 \right\}, \\
\mathcal{N}_{1,3}^{\mathbb{C}} &= \left\{ \begin{bmatrix} 1 & a_{12} & 0 & b_{12} \\ 0 & a_{22} & -1 & b_{22} \end{bmatrix} : a_{i2}, b_{i2} \in \mathbb{C}, i = 1, 2 \right\}, \\
\mathcal{N}_{1,4}^{\mathbb{C}} &= \left\{ \begin{bmatrix} 1 & a_{12} & b_{11} & 0 \\ 0 & a_{22} & b_{21} & 1 \end{bmatrix} : a_{i2}, b_{i1} \in \mathbb{C}, i = 1, 2 \right\}, \\
\mathcal{N}_{2,3}^{\mathbb{C}} &= \left\{ \begin{bmatrix} a_{11} & -1 & 0 & b_{12} \\ a_{21} & 0 & -1 & b_{22} \end{bmatrix} : a_{i1}, b_{i2} \in \mathbb{C}, i = 1, 2 \right\}, \\
\mathcal{N}_{2,4}^{\mathbb{C}} &= \left\{ \begin{bmatrix} a_{11} & -1 & b_{11} & 0 \\ a_{21} & 0 & b_{21} & 1 \end{bmatrix} : a_{i1}, b_{i1} \in \mathbb{C}, i = 1, 2 \right\}, \\
\mathcal{N}_{3,4}^{\mathbb{C}} &= \left\{ \begin{bmatrix} a_{11} & a_{12} & -1 & 0 \\ a_{21} & a_{22} & 0 & -1 \end{bmatrix} : a_{ij} \in \mathbb{C}, i, j = 1, 2 \right\},
\end{aligned} \tag{2.2}$$

which are the so-called canonical coordinate systems on $\mathcal{A}^{\mathbb{C}}$. The map

$$\begin{aligned}
\varphi_{1,2} : \mathcal{N}_{1,2}^{\mathbb{C}} &\rightarrow \mathbb{C}^4, \\
\begin{bmatrix} 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{bmatrix} &\mapsto (b_{11}, b_{12}, b_{21}, b_{22}),
\end{aligned}$$

is homeomorphic, and the coefficient matrix of the BC \mathbf{A} can be written as the form

$$\begin{pmatrix} 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{pmatrix},$$

which is called the corresponding normalized form. There are similar statements about $\mathcal{N}_{1,3}^{\mathbb{C}}$, $\mathcal{N}_{1,4}^{\mathbb{C}}$, $\mathcal{N}_{2,3}^{\mathbb{C}}$, $\mathcal{N}_{2,4}^{\mathbb{C}}$, $\mathcal{N}_{3,4}^{\mathbb{C}}$. One of $\varphi_{i,j}$, $i < j$, $1 \leq i \leq 3$, $2 \leq j \leq 4$, is called a coordinate chart on $\mathcal{A}^{\mathbb{C}}$. The above discussion gives a differentiable structure on $\mathcal{A}^{\mathbb{C}}$. In addition, the space $\mathcal{A}^{\mathbb{R}}$ has a similar atlas of canonical coordinate systems, given by (2.2) with \mathbb{C} replaced by \mathbb{R} everywhere. Using the canonical coordinate systems on $\mathcal{A}^{\mathbb{C}}$ and $\mathcal{A}^{\mathbb{R}}$, it is easy to determine how close to each other any two given BCs are.

For a point p in a differential manifold M , we denote by $T_p M$ the tangent space of M at p . Now we give descriptions of the tangent spaces of $\mathcal{A}^{\mathbb{C}}$ and $\mathcal{A}^{\mathbb{R}}$. If $\mathbf{A} \in \mathcal{N}_{1,2}^{\mathbb{C}}$, then

$$T_{\mathbf{A}} \mathcal{A}^{\mathbb{C}} = T_{\mathbf{A}} \mathcal{N}_{1,2}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & 0 & l_{11} & l_{12} \\ 0 & 0 & l_{21} & l_{22} \end{pmatrix} : l_{ij} \in \mathbb{C}, i, j = 1, 2 \right\}; \tag{2.3}$$

if $\mathbf{A} \in \mathcal{N}_{1,3}^{\mathbb{C}}$, then

$$T_{\mathbf{A}} \mathcal{A}^{\mathbb{C}} = T_{\mathbf{A}} \mathcal{N}_{1,3}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & h_{12} & 0 & l_{12} \\ 0 & h_{22} & 0 & l_{22} \end{pmatrix} : h_{i2}, l_{i2} \in \mathbb{C}, i = 1, 2 \right\}; \tag{2.4}$$

etc. The tangent spaces of $\mathcal{A}^{\mathbb{R}}$ have similar descriptions.

Definition 2.1. (i) A BC $[A | B]$ is said to be self-adjoint if

$$AEA^* = BEB^*,$$

where

$$E := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and A^* denotes the complex conjugate transpose of A . We use $\mathcal{B}^{\mathbb{C}}$ to denote the space of self-adjoint BCs.

(ii) A BC is said to be degenerated if it can be written as the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(iii) A BC is said to be separated if it can be written as the form

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & b_{21} & b_{22} \end{bmatrix}.$$

We use \mathcal{B}_S to denote the space of separated self-adjoint BCs.

(iv) A BC is said to be coupled if it is neither separated nor degenerated.

Remark 2.1. Note that (A, B) satisfies (2.5) if and only if $(A_1, B_1) := (TA, TB)$ does, where $T \in GL(2, \mathbb{C})$. Therefore, the self-adjointness is well-defined. Moreover, the definition of self-adjointness is equivalent to Definition 2.1 in [23].

The following result gives the canonical forms of separated and coupled self-adjoint BCs, respectively. We refer to Theorem 10.4.3 in [27] for details.

Lemma 2.1. *The separated self-adjoint BCs can be written as*

$$\mathbf{S}_{\alpha, \beta} := \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 & 0 \\ 0 & 0 & \cos \beta & -\sin \beta \end{bmatrix}, \quad (2.5)$$

where

$$\alpha \in [0, \pi), \beta \in (0, \pi];$$

and the coupled self-adjoint BCs can be written as

$$[e^{i\gamma} K | -I],$$

where

$$\gamma \in [0, \pi), \quad K \in SL(2, \mathbb{R}) := \{2 \times 2 \text{ real matrix } M : \det M = 1\}.$$

Remark 2.2. The coupled self-adjoint BCs together form an open subset of $\mathcal{B}^{\mathbb{C}}$ and

$$\{[e^{i\gamma} K | -I] : \gamma \in [0, \pi), K \in SL(2, \mathbb{R})\} = \mathcal{N}_{3,4}^{\mathbb{C}} \cap \mathcal{B}^{\mathbb{C}}.$$

The following result gives the topology and the geometric structure of $\mathcal{B}^{\mathbb{C}}$.

Theorem 2.2. *The space $\mathcal{B}^{\mathbb{C}}$ equals the union of the following relative open subsets:*

$$\begin{aligned}
 \mathcal{O}_{1,3}^{\mathbb{C}} &= \left\{ \begin{bmatrix} 1 & a_{12} & 0 & \bar{z} \\ 0 & z & -1 & b_{22} \end{bmatrix} : a_{12}, b_{22} \in \mathbb{R}, z \in \mathbb{C} \right\}, \\
 \mathcal{O}_{1,4}^{\mathbb{C}} &= \left\{ \begin{bmatrix} 1 & a_{12} & \bar{z} & 0 \\ 0 & z & b_{21} & 1 \end{bmatrix} : a_{12}, b_{21} \in \mathbb{R}, z \in \mathbb{C} \right\}, \\
 \mathcal{O}_{2,3}^{\mathbb{C}} &= \left\{ \begin{bmatrix} a_{11} & -1 & 0 & \bar{z} \\ z & 0 & -1 & b_{22} \end{bmatrix} : a_{11}, b_{22} \in \mathbb{R}, z \in \mathbb{C} \right\}, \\
 \mathcal{O}_{2,4}^{\mathbb{C}} &= \left\{ \begin{bmatrix} a_{11} & -1 & \bar{z} & 0 \\ z & 0 & b_{21} & 1 \end{bmatrix} : a_{11}, b_{21} \in \mathbb{R}, z \in \mathbb{C} \right\}.
 \end{aligned} \tag{2.6}$$

Moreover, $\mathcal{B}^{\mathbb{C}}$ is a connected and compact real-analytic manifold of dimension 4.

Proof. Direct calculations yield that all BCs in $\mathcal{O}_{1,3}^{\mathbb{C}}$, $\mathcal{O}_{1,4}^{\mathbb{C}}$, $\mathcal{O}_{2,3}^{\mathbb{C}}$ and $\mathcal{O}_{2,4}^{\mathbb{C}}$ are self-adjoint. Evidently, all separated self-adjoint BCs are in these subsets by Lemma 2.1. Moreover, by Lemma 3.18 in [18], every coupled self-adjoint BC also lies in these subsets. Thus, $\mathcal{B}^{\mathbb{C}}$ is the union of these subsets.

As a canonical coordinate system on $\mathcal{A}^{\mathbb{C}}$, $\mathcal{N}_{1,3}^{\mathbb{C}}$ is open in $\mathcal{A}^{\mathbb{C}}$. It can be easily verified by a direct calculation that

$$\mathcal{O}_{1,3}^{\mathbb{C}} = \mathcal{N}_{1,3}^{\mathbb{C}} \cap \mathcal{B}^{\mathbb{C}},$$

and hence $\mathcal{O}_{1,3}^{\mathbb{C}}$ is a relatively open set in $\mathcal{B}^{\mathbb{C}}$. Similarly, it can be shown that $\mathcal{O}_{1,4}^{\mathbb{C}}$, $\mathcal{O}_{2,3}^{\mathbb{C}}$ and $\mathcal{O}_{2,4}^{\mathbb{C}}$ are relatively open sets in $\mathcal{B}^{\mathbb{C}}$. Since each of $\mathcal{O}_{1,3}^{\mathbb{C}}$, $\mathcal{O}_{1,4}^{\mathbb{C}}$, $\mathcal{O}_{2,3}^{\mathbb{C}}$ and $\mathcal{O}_{2,4}^{\mathbb{C}}$ is connected and intersects the other three, $\mathcal{B}^{\mathbb{C}}$ is connected. The proof of the rest part is the same as that of Theorem 3.11 in [15]. \square

Remark 2.3. (i) There are similar statements (except the dimension) about the space $\mathcal{B}^{\mathbb{R}}$ of real self-adjoint BCs to those about $\mathcal{B}^{\mathbb{C}}$ in Theorem 2.2. Note that $\mathcal{B}^{\mathbb{R}}$ has dimension 3 and is a submanifold of $\mathcal{A}^{\mathbb{R}}$ (see Theorem 3.9 in [15] for detailed discussion).

(ii) Theorem 2.2 says that $\mathcal{O}_{1,3}^{\mathbb{C}}$, $\mathcal{O}_{1,4}^{\mathbb{C}}$, $\mathcal{O}_{2,3}^{\mathbb{C}}$, and $\mathcal{O}_{2,4}^{\mathbb{C}}$ together form an atlas of local coordinate systems on $\mathcal{B}^{\mathbb{C}}$. If $\mathbf{A} \in \mathcal{O}_{1,3}^{\mathbb{C}}$, then the corresponding coordinate chart is given by

$$\hat{\varphi} : \begin{bmatrix} 1 & a_{12} & 0 & \bar{z} \\ 0 & z & -1 & b_{22} \end{bmatrix} \rightarrow (a_{12}, a, b, b_{22}),$$

where $z = a + ib$, $a, b \in \mathbb{R}$. The normalized form of the coefficient matrix of a BC in $\mathcal{O}_{1,3}^{\mathbb{C}}$ is given naturally. Others are given similarly. The above discussion gives a differentiable structure on $\mathcal{B}^{\mathbb{C}}$. There is a similar statement about $\mathcal{B}^{\mathbb{R}}$.

(iii) This result has been first mentioned in Theorem 3.11 in [5], and a proof can be deduced from it.

- (iv) The tangent spaces of $\mathcal{B}^{\mathbb{C}}$ and $\mathcal{B}^{\mathbb{R}}$ can be similarly described as in (2.3), (2.4), etc. Here we omit the details, but they are mentioned in Theorem 4.5 in Section 4 about $\mathcal{B}^{\mathbb{C}}$.

Definition 2.2. The discrete SLP consisting of a discrete SLE $(1/f, q, w)$ and a BC \mathbf{A} is said to be self-adjoint if $(1/f, q, w)$ is in $\Omega_N^{\mathbb{R},+}$ and \mathbf{A} is self-adjoint.

From the above discussions, we immediately deduce the following conclusions, which give the geometric structures of the spaces of discrete SLPs and self-adjoint discrete SLPs, respectively.

Theorem 2.3. *The space $\Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$ of discrete SLPs is a connected complex manifold of complex dimension $3N + 5$, while the space $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ of self-adjoint discrete SLPs is a real-analytic manifold of dimension $3N + 5$ and has 2^{N+1} connected components.*

Remark 2.4. Note that the differentiable structure of the product space $\Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$ can be given by that of $\Omega_N^{\mathbb{C}}$ and $\mathcal{A}^{\mathbb{C}}$ naturally. There is a similar statement about $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$.

3. Multiplicity of eigenvalues and continuous eigenvalue branches

In this section, we shall first discuss properties and relationships of analytic and geometric multiplicities of eigenvalues and then study continuous dependence of eigenvalues on problems. We shall point out that these relationships of the multiplicities of eigenvalues are very important in the following investigations because continuous eigenvalue branches are defined according to the analytic multiplicity of eigenvalues (see Theorem 3.5 and Remark 3.2), while the study on their properties, such as differentiability and monotonicity, is related to the geometric multiplicity of eigenvalues (see Section 4). These relationships allow us to simplify the discussion about the above properties.

3.1. Multiplicity of eigenvalues

In this subsection, we shall first study properties of geometric and analytic multiplicities of eigenvalues of discrete SLPs, separately, and then establish their relationships. Especially, we shall show that they are equal by a direct method if the problem is self-adjoint.

Let l denote the following natural difference operator corresponding to equation (1.1):

$$(ly)_n = \omega_n^{-1} (-\nabla(f_n \Delta y_n) + q_n y_n), \quad n \in [1, N],$$

and let

$$l[0, N+1] = \{y = \{y_n\}_{n=0}^{N+1} : y_n \in \mathbb{C}, 0 \leq n \leq N+1\}.$$

- Definition 3.1.** (i) A complex number λ is called an eigenvalue of the discrete SLP (1.1)-(1.2) if there exists $y \in l[0, N+1]$ which is non-trivial and solves the problem. The non-trivial solution y is called an eigenfunction corresponding to λ .
- (ii) The complex vector space spanned by the eigenfunctions for an eigenvalue is called the eigenspace corresponding to the eigenvalue, while the dimension of the eigenspace is called the geometric multiplicity of the eigenvalue.
- (iii) An eigenfunction $y \in l[0, N+1]$ corresponding to an eigenvalue of the self-adjoint discrete SLP is said to be normalized provided that

$$\sum_{n=1}^N \omega_n y_n \bar{y}_n = 1.$$

- Remark 3.1.* (i) A solution y of (1.1) is said to be non-trivial if there exists $n \in [0, N+1]$ such that $y_n \neq 0$.
- (ii) Since (1.1) has exactly 2 linearly independent solutions, the geometric multiplicity of each eigenvalue is either 1 or 2.

The following uniqueness of solutions of initial value problems of (1.1) can be easily verified.

Lemma 3.1. Let $m \in [0, N]$, and $z_m, z_m^{[1]} \in \mathbb{C}$. Then, for each $\lambda \in \mathbb{C}$, the discrete SLE (1.1) has a unique solution $y(\lambda) \in l[0, N+1]$ satisfying

$$y_m(\lambda) = z_m, \quad f_m \Delta y_m(\lambda) = z_m^{[1]}.$$

Moreover, for each $n \in [0, N]$, $y_n(\lambda)$ and $f_n \Delta y_n(\lambda)$ are polynomials of λ .

For each $\lambda \in \mathbb{C}$, let $\phi(\lambda)$ and $\psi(\lambda)$ be the solutions of (1.1) satisfying the initial conditions, respectively,

$$\phi_0(\lambda) = 1, f_0 \Delta \phi_0(\lambda) = 0; \quad \psi_0(\lambda) = 0, f_0 \Delta \psi_0(\lambda) = 1. \quad (3.1)$$

Then, by Lemma 3.1 any solution of (1.1) is a linear combination of $\phi(\lambda)$ and $\psi(\lambda)$. Set

$$\Phi_n(\lambda) = \begin{pmatrix} \phi_n(\lambda) & \psi_n(\lambda) \\ f_n \Delta \phi_n(\lambda) & f_n \Delta \psi_n(\lambda) \end{pmatrix}, \quad n \in [0, N], \quad \lambda \in \mathbb{C}. \quad (3.2)$$

Equation (1.1) can be rewritten as

$$f_n \Delta y_n = [1 + (q_n - \lambda w_n)/f_{n-1}] f_{n-1} \Delta y_{n-1} + (q_n - \lambda w_n) y_{n-1}, \quad n \in [1, N].$$

So we have

$$\Phi_n(\lambda) = \begin{pmatrix} 1 & 1/f_{n-1} \\ q_n - \lambda w_n & 1 + (q_n - \lambda w_n)/f_{n-1} \end{pmatrix} \Phi_{n-1}(\lambda), \quad n \in [1, N]. \quad (3.3)$$

$\Phi_n(\lambda)$ is called the transfer matrix of (1.1). By induction from (3.3), the leading terms of $\phi_N(\lambda)$, $\psi_N(\lambda)$, $f_N \Delta \phi_N(\lambda)$, and $f_N \Delta \psi_N(\lambda)$ as polynomials

of λ are

$$\begin{aligned} & (-1)^{N-1} \left(\prod_{i=1}^{N-1} (w_i/f_i) \right) \lambda^{N-1}, \quad (-1)^{N-1} \left((1/f_0) \prod_{i=1}^{N-1} (w_i/f_i) \right) \lambda^{N-1}, \\ & (-1)^N \left(w_N \prod_{i=1}^{N-1} (w_i/f_i) \right) \lambda^N, \quad (-1)^N \left((w_N/f_0) \prod_{i=1}^{N-1} (w_i/f_i) \right) \lambda^N, \end{aligned} \quad (3.4)$$

respectively. It follows from (3.1) and (3.3) that

$$\det \Phi_n(\lambda) = 1, \quad n \in [0, N]. \quad (3.5)$$

The following result says that the transfer matrix $\Phi_N(\lambda)$ determines the eigenvalues of the problem for every BC.

Lemma 3.2. *A number $\lambda \in \mathbb{C}$ is an eigenvalue of the discrete SLP (1.1)-(1.2) if and only if λ is a zero of the polynomial*

$$\Gamma(\lambda) := \det(A + B\Phi_N(\lambda)). \quad (3.6)$$

Therefore, either all the complex numbers are eigenvalues of the problem or the problem has only finitely many eigenvalues.

Proof. Fix a $\lambda \in \mathbb{C}$. Let $y(\lambda) := c_1\phi(\lambda) + c_2\psi(\lambda)$, where $c_1, c_2 \in \mathbb{C}$. Inserting $y(\lambda)$ into the BC (1.2) yields

$$(A + B\Phi_N(\lambda)) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0. \quad (3.7)$$

Then $y(\lambda)$ is a non-trivial solution of (1.1) and (1.2), and hence λ is an eigenvalue of the SLP if and only if the determinant of the coefficient matrix in (3.7) vanishes; that is, $\Gamma(\lambda) = 0$. Moreover, by Lemma 3.1 $\Gamma(\lambda)$ is a polynomial of λ . This completes the proof. \square

Definition 3.2. (i) The polynomial function Γ , unique up to a non-zero constant multiple, is called the characteristic function of the discrete SLP, for its importance.

(ii) The order of an eigenvalue as a zero of Γ is called the analytic multiplicity (or simply just multiplicity) of the eigenvalue. An eigenvalue is said to be simple if it has multiplicity 1, while an eigenvalue of multiplicity 2 is called a double eigenvalue.

The following result can be easily deduced from (3.5) and (3.6) via direct calculations. It is useful in some situations.

Lemma 3.3. *The characteristic function of the discrete SLP (1.1)-(1.2) can be written as*

$$\Gamma(\lambda) = \det A + \det B + G(\lambda),$$

where

$$G(\lambda) := c_{11}\phi_N(\lambda) + c_{12}\psi_N(\lambda) + c_{21}f_N\Delta\phi_N(\lambda) + c_{22}f_N\Delta\psi_N(\lambda),$$

$$C := \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}.$$

Theorem 3.1. *For each $\lambda \in \mathbb{C}$, among all boundary conditions, $[\Phi_N(\lambda) \mid -I]$ is the unique one that has λ as an eigenvalue of geometric multiplicity 2.*

Proof. From (3.7), a complex BC $[A \mid B]$ has λ as an eigenvalue of geometric multiplicity 2 if and only if $A = -B\Phi_N(\lambda)$. It follows that

$$(A, B) = (-B(\Phi_N(\lambda), -I). \quad (3.8)$$

Since $\text{rank}(A, B) \leq \text{rank} B$ from (3.8), B is nonsingular by (1.4). Thus, the only BC that has λ as an eigenvalue of geometric multiplicity 2 is the one $[\Phi_N(\lambda) \mid -I]$. The proof is complete. \square

Now, we discuss relationships between the analytic and geometric multiplicities of an eigenvalue of an SLP.

Theorem 3.2. *The analytic multiplicity of an eigenvalue is greater than or equal to its geometric multiplicity.*

Proof. It suffices to prove that the analytic multiplicity of any eigenvalue λ_* of geometric multiplicity 2 is at least 2 by (ii) of Remark 3.1. By Theorem 3.1, we only need to show that as an eigenvalue for the BC $[\Phi_N(\lambda_*) \mid -I]$, λ_* has multiplicity ≥ 2 . Now, in this case it follows from Lemma 3.3 that the characteristic function is given by

$$\begin{aligned} \Gamma(\lambda) &= 2 - (f_N \Delta \psi_N(\lambda_*)) \phi_N(\lambda) + (f_N \Delta \phi_N(\lambda_*)) \psi_N(\lambda) \\ &\quad + \psi_N(\lambda_*) (f_N \Delta \phi_N(\lambda)) - \phi_N(\lambda_*) (f_N \Delta \psi_N(\lambda)). \end{aligned} \quad (3.9)$$

By (3.5) we obtain that

$$\begin{aligned} &\phi'_N(\lambda) (f_N \Delta \psi_N(\lambda)) + \phi_N(\lambda) (f_N \Delta \psi'_N(\lambda)) \\ &- \psi'_N(\lambda) (f_N \Delta \phi_N(\lambda)) - \psi_N(\lambda) (f_N \Delta \phi'_N(\lambda)) \equiv 0, \quad \lambda \in \mathbb{C}. \end{aligned} \quad (3.10)$$

Then, (3.9) and (3.10) together yield that $\Gamma'(\lambda_*) = 0$; that is, the analytic multiplicity of λ_* is at least 2. The proof is complete. \square

We shall remark that the analytic and geometric multiplicities of an eigenvalue are not necessarily equal for an SLP in general, see Examples 5.1 and 5.2. However, we shall show that they are equal in the case that the SLP is self-adjoint.

Next, we consider the self-adjoint case. The self-adjoint SLP (1.1)-(1.2) can be written as (1.5)-(1.6) by setting $d = 1, C_n = f_n, B_n = q_n$,

$$R = (R_1, R_2) = \begin{pmatrix} -a_{11} & b_{11} \\ -a_{21} & b_{21} \end{pmatrix}, \quad S = (S_1, S_2) = \begin{pmatrix} a_{12} & b_{12} \\ a_{22} & b_{22} \end{pmatrix}.$$

Then

$$r := \text{rank}(R_1 + S_1 C_0, S_2) = \text{rank} \begin{pmatrix} -a_{11} + f_0 a_{12} & b_{12} \\ -a_{21} + f_0 a_{22} & b_{22} \end{pmatrix}. \quad (3.11)$$

Obviously, $0 \leq r \leq 2$. Further, we have

$$\kappa := \det(R_1 + S_1 C_0, S_2) = a_{21} b_{12} - a_{11} b_{22} + f_0 (a_{12} b_{22} - a_{22} b_{12}). \quad (3.12)$$

The following result is a direct consequence of Theorem 4.1 in [23].

Lemma 3.4. *The sum of geometric multiplicities of all the eigenvalues of a self-adjoint SLP (1.1)-(1.2) is $N - 2 + r$, and moreover, all its eigenvalues are real.*

The following result can be deduced from Theorem 4.3 in [24]. We shall give an alternative and direct proof as follows.

Theorem 3.3. *The analytic and geometric multiplicities of each eigenvalue of a self-adjoint SLP (1.1)-(1.2) are the same.*

Proof. For convenience, by τ_1 and τ_2 denote the sum of the analytic multiplicities and that of the geometric multiplicities of all the eigenvalues of the self-adjoint SLP, respectively, by $\lambda_1, \dots, \lambda_s$ denote the distinct eigenvalues of the SLP and by $\tau_1^1, \dots, \tau_1^s$ and $\tau_2^1, \dots, \tau_2^s$ denote the corresponding analytic and geometric multiplicities, respectively, where $\tau_1^1 + \dots + \tau_1^s = \tau_1$, $\tau_2^1 + \dots + \tau_2^s = \tau_2$.

The rest proof is divided into two steps.

Step 1. $\tau_1 = \tau_2$.

We divide the discussion into three cases.

Case 1. $r = 2$.

By Lemma 3.4, $\tau_2 = N$. From (3.11), we get that $\kappa \neq 0$. By Lemma 3.3, (3.4), and (3.12) one can get that the leading term of $\Gamma(\lambda)$ as a polynomial of λ is

$$(-1)^{N+1} \left((w_N/f_0) \prod_{i=1}^{N-1} (w_i/f_i) \right) \kappa \lambda^N,$$

and then $\tau_1 = N$. Hence, $\tau_1 = \tau_2 = N$.

Case 2. $r = 1$.

By Lemma 3.4, $\tau_2 = N - 1$. It follows from (3.11) that $\kappa = 0$, and then

$$(-1)^{N+1} \left((w_N/f_0) \prod_{i=1}^{N-1} (w_i/f_i) \right) \kappa \lambda^N = 0.$$

Thus, $\tau_1 \leq N - 1$ by Definition 3.2. Further, we have that $\tau_1 \geq \tau_2$ by Theorem 3.2. Hence, $\tau_1 = \tau_2 = N - 1$.

Case 3. $r = 0$.

By Lemma 3.4, $\tau_2 = N - 2$. From (3.11), we get that

$$a_{11} = f_0 a_{12}, \quad a_{21} = f_0 a_{22}, \quad b_{12} = b_{22} = 0. \quad (3.13)$$

By inserting (3.13) into (1.2) and by (1.4), the BC can be written as the form

$$y_1 = 0, \quad y_N = 0.$$

This implies that there exists a $T \in GL(2, \mathbb{C})$ such that

$$T(A, B) = (A_1, B_1),$$

where

$$A_1 = \begin{pmatrix} f_0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

By Lemma 3.3 we get that

$$\Gamma_1(\lambda) := \det(A_1 + B_1 \Phi_N(\lambda)) = -\phi_N(\lambda) + f_0 \psi_N(\lambda).$$

Then we have that

$$\Gamma(\lambda) = \det T \cdot \Gamma_1(\lambda) = (\det T)(-\phi_N(\lambda) + f_0 \psi_N(\lambda)),$$

which, together with (3.4), implies that the coefficients of the terms λ^N and λ^{N-1} of $\Gamma(\lambda)$ are equal to zero. Thus, $\tau_1 \leq N - 2$ by Definition 3.2. Again by Theorem 3.2 we have that $\tau_1 \geq \tau_2$. Hence, $\tau_1 = \tau_2 = N - 2$.

Step 2. $\tau_1^i = \tau_2^i$ for $1 \leq i \leq s$.

By Theorem 3.2, $\tau_1^i \geq \tau_2^i$ for $1 \leq i \leq s$. Suppose that there exists a j , $1 \leq j \leq s$, such that $\tau_1^j > \tau_2^j$. Then

$$\tau_1 = \sum_{i=1}^s \tau_1^i > \sum_{i=1}^s \tau_2^i = \tau_2, \quad (3.14)$$

which contradicts to $\tau_1 = \tau_2$. Therefore, the assertion holds and the entire proof is complete. \square

The following result is a direct consequence of Theorems 3.1 and 3.3. It has been firstly given in Theorem 4.3.1 in [1] for a special class of separated self-adjoint boundary conditions and then in Theorem 2.2.6 in [12] for more general case.

Corollary 3.1. *Assume that (1.1) is in $\Omega_N^{\mathbb{R},+}$. Then all the eigenvalues for each separated self-adjoint boundary condition are simple.*

3.2. Continuous eigenvalue branches

In this subsection, using the topologies and geometric structures on the space of discrete SLPs introduced in Section 2, we shall show that sufficiently close discrete SLPs have near-by eigenvalues in a given bounded region of \mathbb{C} , and explain how such eigenvalues form the so-called continuous eigenvalue branches. In a general case, all the simple eigenvalues live in so-called continuous simple eigenvalue branches over the space of problems. However, we can get a better result in the self-adjoint case that all the eigenvalues, which may be simple or not simple, live in continuous eigenvalue branches over the space of the problems.

To indicate the dependence of $\Phi_n(\lambda)$ on the discrete SLE (1.1), we write $\Phi_n(\lambda, \omega)$ with $\omega = (1/f, q, w) \in \Omega_N^{\mathbb{C}}$. The following result can be deduced from (3.1) and (3.3).

Lemma 3.5. *Let $\omega \in \Omega_N^{\mathbb{C}}$. For each $\varepsilon > 0$, there is $\delta > 0$ such that if $\sigma \in \Omega_N^{\mathbb{C}}$ satisfies $\|\sigma - \omega\| < \delta$, then*

$$\|\Phi_n(\lambda, \sigma) - \Phi_n(\lambda, \omega)\|_1 < \varepsilon, \quad n \in [0, N], \quad |\lambda| \leq 1/\varepsilon,$$

where $\|\cdot\|_1$ is the maximum norm for the 2×2 matrix.

Now, we are ready to prove the locally continuous dependence of eigenvalues on the corresponding discrete SLP.

Theorem 3.4. *Let $\lambda_* \in \mathbb{C}$ be an eigenvalue of an SLP $(\omega, \mathbf{A}) \in \Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$ with multiplicity m , R a bounded open subset of \mathbb{C} such that $\lambda_* \in R$, and λ_* the only eigenvalue of (ω, \mathbf{A}) in the closure \bar{R} of R . Then, there is a neighborhood \mathcal{U} of (ω, \mathbf{A}) in $\Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$ such that each problem in \mathcal{U} has exactly m eigenvalues in \bar{R} , counting multiplicity, and they all lie in R .*

Proof. To indicate the dependence of $\Gamma(\lambda)$ on the SLP (ω, \mathbf{A}) , we write $\Gamma_{(\omega, \mathbf{A})}(\lambda)$. Let \mathcal{N} be a coordinate system in (2.2) containing \mathbf{A} . For all BCs in \mathcal{N} , we compute the characteristic function using the corresponding normalized form of the coefficient matrices of the BCs. By Lemma 3.5, when $(\sigma, \mathbf{B}) \in \Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$ is sufficiently close to (ω, \mathbf{A}) , \mathbf{B} is also in \mathcal{N} , and $\Gamma_{(\sigma, \mathbf{B})}(\lambda)$ is close to $\Gamma_{(\omega, \mathbf{A})}(\lambda)$ on \bar{R} . Since $\Gamma_{(\omega, \mathbf{A})}(\lambda)$ (or $\Gamma_{(\sigma, \mathbf{B})}(\lambda)$) is a polynomial of λ and the degree of $\Gamma_{(\omega, \mathbf{A})}(\lambda)$ (or $\Gamma_{(\sigma, \mathbf{B})}(\lambda)$) in λ is less than or equal to N by Lemma 3.3 and (3.4), we can set

$$\Gamma_{(\omega, \mathbf{A})}(\lambda) = a_k \lambda^k + \cdots + a_1 \lambda + a_0,$$

$\Gamma_{(\sigma, \mathbf{B})}(\lambda) = a_k \lambda^k + \cdots + a_1 \lambda + a_0 + \varepsilon_N(\sigma, \mathbf{B}) \lambda^N + \cdots + \varepsilon_1(\sigma, \mathbf{B}) \lambda + \varepsilon_0(\sigma, \mathbf{B})$, where $k \leq N$, (σ, \mathbf{B}) is sufficiently close to (ω, \mathbf{A}) , and the value of $\varepsilon_i(\sigma, \mathbf{B}) \in \mathbb{C}$ is dependent on $(\sigma, \mathbf{B}) \in \Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$, $0 \leq i \leq N$. Since the boundary set $\partial \bar{R}$ is a compact subset of \mathbb{C} and $\Gamma_{(\omega, \mathbf{A})}(\lambda) \neq 0$ for all $\lambda \in \partial \bar{R}$, there exists $\lambda_0 \in \partial \bar{R}$ such that

$$\inf_{\lambda \in \partial \bar{R}} |\Gamma_{(\omega, \mathbf{A})}(\lambda)| = |\Gamma_{(\omega, \mathbf{A})}(\lambda_0)| =: \eta > 0. \quad (3.15)$$

One can choose $\varepsilon > 0$ satisfying that

$$\varepsilon \cdot \sup_{\lambda \in \partial \bar{R}} (|\lambda|^N + \cdots + |\lambda| + 1) < \eta. \quad (3.16)$$

Since $\Gamma_{(\sigma, \mathbf{B})}(\lambda) \rightarrow \Gamma_{(\omega, \mathbf{A})}(\lambda)$ uniformly for $\lambda \in \bar{R}$ as $(\sigma, \mathbf{B}) \rightarrow (\omega, \mathbf{A})$, there exists a neighborhood \mathcal{U} of (ω, \mathbf{A}) in $\Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$ such that

$$|\varepsilon_i(\sigma, \mathbf{B})| < \varepsilon, \quad 0 \leq i \leq N,$$

which, together with (3.16), yields that

$$|\varepsilon_N(\sigma, \mathbf{B}) \lambda^N + \cdots + \varepsilon_1(\sigma, \mathbf{B}) \lambda + \varepsilon_0(\sigma, \mathbf{B})| < \eta, \quad \lambda \in \partial \bar{R}.$$

Therefore, we have by (3.15) that

$$|\Gamma_{(\omega, \mathbf{A})}(\lambda)| > |\varepsilon_N(\sigma, \mathbf{B}) \lambda^N + \cdots + \varepsilon_1(\sigma, \mathbf{B}) \lambda + \varepsilon_0(\sigma, \mathbf{B})|, \quad \lambda \in \partial \bar{R}.$$

By Rouché's Theorem in complex analysis, $\Gamma_{(\sigma, \mathbf{B})}(\lambda)$ and $\Gamma_{(\omega, \mathbf{A})}(\lambda)$ have the same number of zeros in R , counting order. Therefore, the SLP $(\sigma, \mathbf{B}) \in \mathcal{U}$ has exactly m eigenvalues in \bar{R} , counting multiplicity, and they all lie in R . This proof is complete. \square

The following result is a direct consequence of Theorem 3.4.

Corollary 3.2. *For each $m \in \mathbb{N}$, the set of discrete SLPs having at least m eigenvalues, counting multiplicity, is open in $\Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$.*

Combining the reality of the eigenvalues for a self-adjoint discrete SLP and Theorem 3.4 yields the following result:

Corollary 3.3. *Let $\lambda_* \in \mathbb{R}$ be an eigenvalue of a discrete SLP $(\omega, \mathbf{A}) \in \Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ with multiplicity m , (r_1, r_2) a bounded open interval of \mathbb{R} such that $\lambda_* \in (r_1, r_2)$, and λ_* the only eigenvalue of (ω, \mathbf{A}) in the close interval $[r_1, r_2]$. Then, there is a neighborhood \mathcal{U} of (ω, \mathbf{A}) in $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ such that each problem in \mathcal{U} has exactly m eigenvalues in $[r_1, r_2]$, counting multiplicity, and they all lie in (r_1, r_2) .*

Based on the above discussion, we now form the continuous eigenvalue branches over $\Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$ or $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ through a fixed eigenvalue.

Theorem 3.5. (1) *Let λ_* be a simple eigenvalue for a discrete SLP $(\omega, \mathbf{A}) \in \Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$. Then there is a continuous function $\Lambda : \mathcal{M} \rightarrow \mathbb{C}$ defined on a connected neighborhood \mathcal{M} of (ω, \mathbf{A}) in $\Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$ such that*

$$(i) \quad \Lambda(\omega, \mathbf{A}) = \lambda_*;$$

$$(ii) \quad \text{for any } (\sigma, \mathbf{B}) \in \mathcal{M}, \Lambda(\sigma, \mathbf{B}) \text{ is a simple eigenvalue of } (\sigma, \mathbf{B}).$$

(2) *Let λ_* be an eigenvalue of a self-adjoint discrete SLP (ω, \mathbf{A}) with multiplicity 2. Fix a small $\epsilon > 0$ such that λ_* is the only eigenvalue of (ω, \mathbf{A}) in the interval $[\lambda_* - \epsilon, \lambda_* + \epsilon]$. Then, there are continuous functions $\Lambda_1, \Lambda_2 : \mathcal{F} \rightarrow \mathbb{R}$ defined on a connected neighborhood \mathcal{F} of (ω, \mathbf{A}) in $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ such that*

$$(i) \quad \Lambda_1(\omega, \mathbf{A}) = \Lambda_2(\omega, \mathbf{A}) = \lambda_*;$$

$$(ii) \quad \lambda_* - \epsilon < \Lambda_1(\sigma, \mathbf{B}) \leq \Lambda_2(\sigma, \mathbf{B}) < \lambda_* + \epsilon \text{ for each } (\sigma, \mathbf{B}) \in \mathcal{F};$$

$$(iii) \quad \text{for every } (\sigma, \mathbf{B}) \in \mathcal{F}, \Lambda_1(\sigma, \mathbf{B}) \text{ and } \Lambda_2(\sigma, \mathbf{B}) \text{ are eigenvalues of } (\sigma, \mathbf{B}).$$

Proof. Assertion (1) can be straightforward shown by Theorem 3.4.

Now, we show assertion (2). By Corollary 3.3 there is a neighborhood \mathcal{F} of (ω, \mathbf{A}) in $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ such that each problem (σ, \mathbf{B}) in \mathcal{F} has exactly 2 eigenvalues, which are denoted by $\lambda_1(\sigma, \mathbf{B})$ and $\lambda_2(\sigma, \mathbf{B})$ with $\lambda_1(\sigma, \mathbf{B}) \leq \lambda_2(\sigma, \mathbf{B})$, respectively, in $[\lambda_* - \epsilon, \lambda_* + \epsilon]$ and they all lie in $(\lambda_* - \epsilon, \lambda_* + \epsilon)$. Note that \mathcal{F} can be chosen such that it belongs to a connected component of $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ by Theorem 2.3. Then one can define two functions $\Lambda_1, \Lambda_2 : \mathcal{F} \rightarrow \mathbb{R}$ such that $\Lambda_1(\sigma, \mathbf{B}) = \lambda_1(\sigma, \mathbf{B})$ and $\Lambda_2(\sigma, \mathbf{B}) = \lambda_2(\sigma, \mathbf{B})$. It can be easily verified that Λ_1 and Λ_2 satisfy (i)-(iii).

Next, we prove that $\Lambda_1, \Lambda_2 : \mathcal{F} \rightarrow \mathbb{R}$ are continuous functions. Fix a $(\sigma, \mathbf{B}) \in \mathcal{F}$. If $\Lambda_1(\sigma, \mathbf{B}) = \Lambda_2(\sigma, \mathbf{B})$, there exists a neighborhood $\mathcal{U}_{(r_1, r_2)}$ of (σ, \mathbf{B}) in $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ such that $\Lambda_1(\tau, \mathbf{C}), \Lambda_2(\tau, \mathbf{C}) \in (r_1, r_2)$ for each open interval (r_1, r_2) satisfying $\Lambda_1(\sigma, \mathbf{B}) \in (r_1, r_2) \subset [\lambda_* - \epsilon, \lambda_* + \epsilon]$ and each $(\tau, \mathbf{C}) \in \mathcal{U}_{(r_1, r_2)}$ by Corollary 3.3. Now, we assume that $\Lambda_1(\sigma, \mathbf{B}) < \Lambda_2(\sigma, \mathbf{B})$. Then, there exists a positive number ϵ_1 such that

$$\bigcup_{i=1}^2 (\Lambda_i(\sigma, \mathbf{B}) - \epsilon_1, \Lambda_i(\sigma, \mathbf{B}) + \epsilon_1) \subset (\lambda_* - \epsilon, \lambda_* + \epsilon)$$

and

$$\bigcap_{i=1}^2 (\Lambda_i(\boldsymbol{\sigma}, \mathbf{B}) - \epsilon_1, \Lambda_i(\boldsymbol{\sigma}, \mathbf{B}) + \epsilon_1) = \emptyset.$$

By Corollary 3.3 there exists a neighborhood \mathcal{U}_δ of $(\boldsymbol{\sigma}, \mathbf{B})$ in $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ such that $\Lambda_1(\boldsymbol{\tau}, \mathbf{C}) \in (\Lambda_1(\boldsymbol{\sigma}, \mathbf{B}) - \delta, \Lambda_1(\boldsymbol{\sigma}, \mathbf{B}) + \delta)$, $\Lambda_2(\boldsymbol{\tau}, \mathbf{C}) \in (\Lambda_2(\boldsymbol{\sigma}, \mathbf{B}) - \delta, \Lambda_2(\boldsymbol{\sigma}, \mathbf{B}) + \delta)$ for each δ satisfying $0 < \delta < \epsilon_1$ and each $(\boldsymbol{\tau}, \mathbf{C}) \in \mathcal{U}_\delta$. Therefore, Λ_1 and Λ_2 are continuous in \mathcal{F} . This completes the proof. \square

Remark 3.2. (1) In the case of general discrete SLP, that is, $(\boldsymbol{\omega}, \mathbf{A}) \in \Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$, any two such functions as Λ in (1) of Theorem 3.5 agree on the common part of their domains, which is still a neighborhood of $(\boldsymbol{\omega}, \mathbf{A})$ in $\Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$. So, by the continuous simple eigenvalue branch over $\Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$ through λ_* , we mean any such function. In the case of self-adjoint discrete SLP, that is, $(\boldsymbol{\omega}, \mathbf{A}) \in \Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$, and λ_* is an eigenvalue with multiplicity 2, there are actually different functions on \mathcal{F} . Locally they are the only such functions, to be called the continuous eigenvalue branches over $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ through λ_* .

- (2) Statement (1) of Theorem 3.5 holds if we replace $\Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$ and \mathbb{C} by $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ and \mathbb{R} , respectively. This gives the continuous simple eigenvalue branch over $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ through λ_* .
- (3) There are similar results for subspaces of $\Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$, such as $\Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{R}}$ and $\Omega_N^{\mathbb{R}} \times \mathcal{A}^{\mathbb{C}}$ to (1) of Theorem 3.5. There are similar results for subspaces of $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$, such as $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{R}}$ to (2) of Theorem 3.5 and (2) of Remark 3.2.
- (4) The third author of the present paper, together with Lv, obtained the continuous dependence of eigenvalues on problems under a non-singularity condition (see Theorem 3.1 and Corollary 3.1 in [17]).

The following result can be directly obtained by Theorem 3.2 and (1) of Theorem 3.5.

Corollary 3.4. *Let λ_* be a simple eigenvalue of a discrete SLP $(\boldsymbol{\omega}, \mathbf{A}) \in \Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$, and Λ the continuous simple eigenvalue branch over $\Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$ through λ_* . Then, there exists a connected neighborhood \mathcal{M} of $(\boldsymbol{\omega}, \mathbf{A})$ such that for each $(\boldsymbol{\sigma}, \mathbf{B}) \in \mathcal{M}$, $\Lambda(\boldsymbol{\sigma}, \mathbf{B})$ has geometric multiplicity 1.*

4. Analyticity, differentiability, and monotonicity

In this section, we shall investigate analyticity and differentiability of continuous eigenvalue branches under some assumptions on their multiplicities, and then study monotonicity of continuous eigenvalue branches of self-adjoint discrete SLPs on boundary conditions and equations, separately.

4.1. Analyticity and differentiability of continuous simple eigenvalue branches

In this subsection, we shall study the analyticity and differentiability of continuous simple eigenvalue branches. To do this, we need the following two lemmas (see Theorem 2.1.2 in [11] and Chapter V in [20], separately):

Lemma 4.1. Let $f_j(w, z)$, $1 \leq j \leq m$, be analytic functions of $(w, z) = (w_1, \dots, w_m, z_1, \dots, z_n)$ in a neighborhood of a point (w^0, z^0) in $\mathbb{C}^m \times \mathbb{C}^n$, and assume that $f_j(w^0, z^0) = 0$, $1 \leq j \leq m$, and

$$\det(\partial f_j / \partial w_k)_{j,k=1}^m \big|_{(w^0, z^0)} \neq 0.$$

Then the equations $f_j(w, z) = 0$, $1 \leq j \leq m$, have a uniquely determined analytic solution $w(z)$ in a neighborhood of z^0 such that $w(z^0) = w^0$. Moreover, the derivative formula in the neighborhood of z^0 is determined by

$$\sum_{k=1}^m (\partial f_j / \partial w_k) dw_k + \sum_{i=1}^n (\partial f_j / \partial z_i) dz_i = 0, \quad 1 \leq j \leq m. \quad (4.1)$$

Lemma 4.2. Assume that $U \subset \mathbb{R}^{n+1}$ is an open set and $F : U \rightarrow \mathbb{R}$ is a C^r function for some $r \geq 1$. For $p \in \mathbb{R}^{n+1}$, we write $p = (x, y)$ with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. Assume that $(x_0, y_0) \in U$ and

$$(\partial F / \partial y)(x_0, y_0) \neq 0.$$

Let $C = F(x_0, y_0) \in \mathbb{R}$. Then, there are open sets V containing x_0 and W containing y_0 with $V \times W \subset U$, and a C^r function $h : V \rightarrow W$ such that $h(x_0) = y_0$ and

$$F(x, h(x)) = C \quad \text{for all } x \in V.$$

Further, for each $x \in V$, $h(x)$ is the unique $y \in W$ such that $F(x, y) = C$.

Theorem 4.1. Let $\lambda_* \in \mathbb{C}$ be a simple eigenvalue of a discrete SLP $(\omega, \mathbf{A}) \in \Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$. Then, the continuous simple eigenvalue branch Λ defined on a neighborhood \mathcal{F} of (ω, \mathbf{A}) in $\Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$ through λ_* is analytic. For a fixed discrete SLE, the derivative of Λ at $\mathbf{A} = [A \mid B]$ is given by

$$d\Lambda|_{\mathbf{A}}(H \mid L) = - \sum_{j,k=1}^2 (d_{jk} h_{jk} + e_{jk} l_{jk}) / G'(\lambda_*), \quad (H \mid L) \in T_{\mathbf{A}} \mathcal{A}^{\mathbb{C}}, \quad (4.2)$$

where the coefficient matrices $D = (d_{jk})$ and $E = (e_{jk})$ are defined by

$$D := \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix} + \begin{pmatrix} b_{22} & -b_{21} \\ -b_{12} & b_{11} \end{pmatrix} \begin{pmatrix} f_N \Delta \psi_N(\lambda_*) & -f_N \Delta \phi_N(\lambda_*) \\ -\psi_N(\lambda_*) & \phi_N(\lambda_*) \end{pmatrix},$$

$$E := \begin{pmatrix} b_{22} & -b_{21} \\ -b_{12} & b_{11} \end{pmatrix} + \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix} \begin{pmatrix} \phi_N(\lambda_*) & f_N \Delta \phi_N(\lambda_*) \\ \psi_N(\lambda_*) & f_N \Delta \psi_N(\lambda_*) \end{pmatrix}.$$

Proof. For the fixed problem (ω, \mathbf{A}) , we assume that

$$\omega = (1/f, q, w) \in \Omega_N^{\mathbb{C}}, \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{bmatrix} \in \mathcal{N}_{1,2}^{\mathbb{C}}.$$

For all BCs in $\mathcal{N}_{1,2}^{\mathbb{C}}$, we compute Γ using the corresponding normalized form of the coefficient matrices of the BCs. Define

$$\tilde{\varphi} : \Omega_N^{\mathbb{C}} \times \mathcal{N}_{1,2}^{\mathbb{C}} \rightarrow \mathbb{C}^{3N+5},$$

$$(1/f', q', w') \times \begin{bmatrix} 1 & 0 & b'_{11} & b'_{12} \\ 0 & 1 & b'_{21} & b'_{22} \end{bmatrix} \mapsto (1/f', q', w', b'_{11}, b'_{12}, b'_{21}, b'_{22}).$$

Then $\tilde{\varphi}$ is a coordinate chart on $\Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$. For convenience, we set

$$\mathcal{V} = \tilde{\varphi} \left((\Omega_N^{\mathbb{C}} \times \mathcal{N}_{1,2}^{\mathbb{C}}) \cap \mathcal{F} \right),$$

$$\mathcal{K} := \{(\lambda, 1/f', q', w', b'_{11}, b'_{12}, b'_{21}, b'_{22}) : \lambda \in \mathbb{C}, (1/f', q', w', b'_{11}, \dots, b'_{22}) \in \mathcal{V}\}.$$

Now, consider Γ restricted to the region \mathcal{K} . By (3.3) and Lemma 3.2, Γ is a polynomial and hence an analytic function of all variables in \mathcal{K} . Since $\lambda_* = \Lambda(\boldsymbol{\omega}, \mathbf{A})$ is simple, we have that

$$\Gamma'(\lambda_*) = G'(\lambda_*) \neq 0. \quad (4.3)$$

By Lemma 4.1 $\Lambda\tilde{\varphi}^{-1}$ is exactly the uniquely determined analytic solution to the equation

$$\Gamma(\lambda) = 0$$

on λ in a neighborhood of $(1/f, q, w, b_{11}, b_{12}, b_{21}, b_{22})$ such that

$$\Lambda\tilde{\varphi}^{-1}(1/f, q, w, b_{11}, b_{12}, b_{21}, b_{22}) = \lambda_*.$$

Hence, Λ is analytic at $(\boldsymbol{\omega}, \mathbf{A})$. If we replace $(\boldsymbol{\omega}, \mathbf{A})$ by $(\boldsymbol{\sigma}, \mathbf{B}) \in \mathcal{F}$, a similar argument above yields that Λ is analytic at $(\boldsymbol{\sigma}, \mathbf{B})$. Therefore, Λ is analytic in the neighborhood \mathcal{F} of $(\boldsymbol{\omega}, \mathbf{A})$.

Fix a discrete SLE. For any $(H | L) \in \mathbf{T}_{\mathbf{A}}\mathcal{A}^{\mathbb{C}}$, applying (4.1) to Γ in a neighborhood of $(\lambda_*, b_{11}, b_{12}, b_{21}, b_{22})$, one can deduce

$$G'(\lambda_*) d(\Lambda\tilde{\varphi}^{-1}) + \sum_{j,k=1}^2 (\partial\Gamma/\partial b_{jk}) db_{jk} = 0,$$

where

$$\left(\frac{\partial\Gamma}{\partial b_{jk}} \right)_{2 \times 2} = \begin{pmatrix} b_{22} & -b_{21} \\ -b_{12} & b_{11} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_N(\lambda_*) & f_N \Delta \phi_N(\lambda_*) \\ \psi_N(\lambda_*) & f_N \Delta \psi_N(\lambda_*) \end{pmatrix},$$

which, together with (4.3), implies (4.2).

If $\mathbf{A} \in \mathcal{N}_{i,j}^{\mathbb{C}}$, where $1 \leq i \leq 3, 3 \leq j \leq 4, i < j$, the proof can be completed by a method analogous to that used above. \square

With the help of Lemma 4.2, one can deduce the following result using the same method above:

Theorem 4.2. *Let $\lambda_* \in \mathbb{R}$ be a simple eigenvalue of a self-adjoint discrete SLP $(\boldsymbol{\omega}, \mathbf{A}) \in \Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$. Then, the continuous simple eigenvalue branch Λ defined on a neighborhood of $(\boldsymbol{\omega}, \mathbf{A})$ in $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ through λ_* is a C^∞ function.*

Remark 4.1. If $\lambda_* \in \mathbb{R}$ is an eigenvalue of $(\boldsymbol{\omega}, \mathbf{A}) \in \Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ with multiplicity 2, the continuous eigenvalue branch Λ defined on a neighborhood of $(\boldsymbol{\omega}, \mathbf{A})$ in $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ through λ_* is not necessarily differentiable. Please see Examples 5.4-5.7 for illustration.

4.2. Monotonicity on boundary conditions of continuous eigenvalue branches of self-adjoint discrete SLPs

In this subsection, we shall investigate monotonicity of continuous simple eigenvalue branches on boundary conditions in several subsets of $\mathcal{B}^{\mathbb{C}}$ for self-adjoint discrete SLPs using the derivative formulas of continuous simple eigenvalue branches with respect to the corresponding BC.

Lemma 4.3. *Let λ_* be a simple eigenvalue of a discrete SLP $(\omega, \mathbf{A}) \in \Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$, and Λ the continuous simple eigenvalue branch defined on a neighborhood \mathcal{U} of (ω, \mathbf{A}) in $\Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$ through λ_* . Then, there is a continuous choice $u(\sigma, \mathbf{B}) \in l[0, N+1]$ of eigenfunction corresponding to $\Lambda(\sigma, \mathbf{B})$ for all $(\sigma, \mathbf{B}) \in \Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$ sufficiently close to (ω, \mathbf{A}) . Here, the continuity of $u(\sigma, \mathbf{B})$ means that for each $(\tau, \mathbf{C}) \in \Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$ sufficiently close to (ω, \mathbf{A}) ,*

$$u(\sigma, \mathbf{B}) \rightarrow u(\tau, \mathbf{C}) \quad \text{in } \mathbb{C}^{N+2}$$

as $(\sigma, \mathbf{B}) \rightarrow (\tau, \mathbf{C})$ in $\Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$.

Proof. Every eigenfunction of the SLP (ω, \mathbf{A}) corresponding to $\Lambda(\omega, \mathbf{A}) = \lambda_*$ can be written as

$$u(\omega, \mathbf{A}) = C_1(\omega, \mathbf{A})\phi(\Lambda(\omega, \mathbf{A})) + C_2(\omega, \mathbf{A})\psi(\Lambda(\omega, \mathbf{A})), \quad (4.4)$$

where $C_1(\omega, \mathbf{A}), C_2(\omega, \mathbf{A}) \in \mathbb{C}$ are dependent on (ω, \mathbf{A}) . Inserting (4.4) into (1.2), we get

$$(A + B\Phi_N(\Lambda(\omega, \mathbf{A}))) \begin{pmatrix} C_1(\omega, \mathbf{A}) \\ C_2(\omega, \mathbf{A}) \end{pmatrix} = 0. \quad (4.5)$$

Set

$$M(\omega, \mathbf{A}) = (m_{ij}(\omega, \mathbf{A}))_{2 \times 2} := A + B\Phi_N(\Lambda(\omega, \mathbf{A})). \quad (4.6)$$

Since λ_* is simple, $\Lambda(\sigma, \mathbf{B})$ is continuous in \mathcal{U} and has geometric multiplicity 1 for each $(\sigma, \mathbf{B}) \in \mathcal{U}$ by Theorem 3.5 and Corollary 3.4, and one has that

$$\text{rank} M(\omega, \mathbf{A}) = 1, \quad (4.7)$$

which implies $m_{i_0 j_0}(\omega, \mathbf{A}) \neq 0$ for some $1 \leq i_0, j_0 \leq 2$. Without loss of generality, we assume that $m_{11}(\omega, \mathbf{A}) \neq 0$. By replacing (ω, \mathbf{A}) with $(\sigma, \mathbf{B}) \in \mathcal{U}$ in (4.4)-(4.7), the similar equations denoted by (4.4') – (4.7') still hold. Obviously, there exists a neighborhood $\hat{\mathcal{U}}$ of (ω, \mathbf{A}) with $\hat{\mathcal{U}} \subset \mathcal{U}$ such that $m_{11}(\sigma, \mathbf{B}) \neq 0$ for each $(\sigma, \mathbf{B}) \in \hat{\mathcal{U}}$. It is evident that

$$C_1(\sigma, \mathbf{B}) = m_{12}(\sigma, \mathbf{B}), \quad C_2(\sigma, \mathbf{B}) = -m_{11}(\sigma, \mathbf{B}),$$

is a solution of (4.5') for each $(\sigma, \mathbf{B}) \in \hat{\mathcal{U}}$ by (4.7'). Hence, $u(\sigma, \mathbf{B})$ defined by (4.4') is an eigenfunction corresponding to $\Lambda(\sigma, \mathbf{B})$ and continuous in $\hat{\mathcal{U}}$ by the fact that $M(\sigma, \mathbf{B})$ is continuous in \mathcal{U} . This completes the proof. \square

Remark 4.2. If $\Omega_N^{\mathbb{C}} \times \mathcal{A}^{\mathbb{C}}$ is replaced by $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ in Lemma 4.3, then it still holds for the self-adjoint discrete SLPs.

By Corollary 3.1, all eigenvalues of every self-adjoint SLP with separated BC are simple. With the help of the preceding lemma we can now give related derivative formulas of continuous simple eigenvalue branch Λ with respect to the parameters of the separated self-adjoint BCs. To indicate the dependence of $\Lambda(\mathbf{S}_{\alpha,\beta})$ on the parameters α and β , we sometimes write $\Lambda(\alpha, \beta) = \Lambda(\mathbf{S}_{\alpha,\beta})$ for $\mathbf{S}_{\alpha,\beta} \in \mathcal{B}_S$.

Theorem 4.3. *Assume that λ_* is an eigenvalue of a self-adjoint discrete SLP $(\omega, \mathbf{S}_{\alpha,\beta})$ with $\mathbf{S}_{\alpha,\beta} \in \mathcal{B}_S$. Let $y \in l[0, N+1]$ be a normalized eigenfunction for λ_* , and Λ the continuous eigenvalue branch over \mathcal{B}_S through λ_* . Then, its derivatives are given by*

$$\Lambda'_\alpha(\alpha, \beta) = -|y_0|^2 - |f_0 \Delta y_0|^2, \quad \Lambda'_\beta(\alpha, \beta) = |y_N|^2 + |f_N \Delta y_N|^2. \quad (4.8)$$

Proof. We first show that the first relation in (4.8) holds. Fix all the components of $(\omega, \mathbf{S}_{\alpha,\beta})$ except α . Let $y = y(\cdot, \alpha)$. By Corollary 3.1, λ_* is a simple eigenvalue of $(\omega, \mathbf{S}_{\alpha,\beta})$. By Remark 4.2, we can choose an eigenfunction $z = y(\cdot, \alpha + h)$ with respect to $\Lambda(\alpha + h, \beta)$ for $h \in \mathbb{R}$ sufficiently small such that $z \rightarrow y$ as $h \rightarrow 0$. From (1.1) we get that

$$(\Lambda(\alpha + h, \beta) - \Lambda(\alpha, \beta)) w_n y_n \bar{z}_n = -\Delta[y_{n-1}, z_{n-1}],$$

where $[y_n, z_n] := y_n(f_n \Delta \bar{z}_n) - (f_n \Delta y_n) \bar{z}_n$. Hence,

$$(\Lambda(\alpha + h, \beta) - \Lambda(\alpha, \beta)) \sum_{n=1}^N w_n y_n \bar{z}_n = [y_0, z_0] - [y_N, z_N]. \quad (4.9)$$

The BC $\mathbf{S}_{\alpha,\beta}$ with respect to β implies that

$$[y_N, z_N] = 0. \quad (4.10)$$

In the case that $\alpha \neq \pi/2$, by the BC $\mathbf{S}_{\alpha,\beta}$ with respect to α , together with (4.9) and (4.10), we get that

$$(\Lambda(\alpha + h, \beta) - \Lambda(\alpha, \beta)) \sum_{n=1}^N w_n y_n \bar{z}_n = -(\tan(\alpha + h) - \tan \alpha)(f_0 \Delta y_0)(f_0 \Delta \bar{z}_0). \quad (4.11)$$

Dividing both sides of (4.11) by h and taking the limit as $h \rightarrow 0$, we obtain that

$$\Lambda'_\alpha(\alpha, \beta) = -|f_0 \Delta y_0|^2 \sec^2 \alpha = -|y_0|^2 - |f_0 \Delta y_0|^2.$$

In the other case that $\alpha = \pi/2$, by the BC $\mathbf{S}_{\pi/2,\beta}$ with respect to α , together with (4.9) and (4.10), we get that $f_0 \Delta y_0 = 0$ and

$$(\Lambda(\pi/2 + h, \beta) - \Lambda(\pi/2, \beta)) \sum_{n=1}^N w_n y_n \bar{z}_n = \cot(\pi/2 + h) y_0 \bar{z}_0. \quad (4.12)$$

Dividing both sides of (4.12) by h and taking the limit as $h \rightarrow 0$, we obtain that

$$\Lambda'_\alpha(\pi/2, \beta) = -|y_0|^2.$$

Hence, the first relation in (4.8) follows.

With a similar argument to that used in the above discussion, one can show that the second relation in (4.8) holds. This completes the proof. \square

The following result is directly derived from Corollary 3.1 and Theorem 4.3.

Theorem 4.4. *Assume that (1.1) is in $\Omega_N^{\mathbb{R},+}$. Then each continuous eigenvalue branch over \mathcal{B}_S is always strictly decreasing in the α -direction and always strictly increasing in the β -direction.*

The following results give the derivative formulas of continuous simple eigenvalue branch Λ over $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ with respect to $\mathcal{B}^{\mathbb{C}}$.

Theorem 4.5. *Fix $\omega \in \Omega_N^{\mathbb{R},+}$. Let λ_* be a simple eigenvalue of (ω, \mathbf{A}) for a self-adjoint boundary condition \mathbf{A} , $y \in l[0, N+1]$ be a normalized eigenfunction for λ_* , and Λ be the continuous simple eigenvalue branch of (ω, \mathbf{B}) for $\mathbf{B} \in \mathcal{B}^{\mathbb{C}}$ through λ_* . Then, we have the following derivative formulas:*

(1) when $\mathbf{A} \in \mathcal{O}_{1,3}^{\mathbb{C}}$,

$$d\Lambda|_{\mathbf{A}}(H|L) = (f_0\Delta\bar{y}_0, f_N\Delta\bar{y}_N) \begin{pmatrix} h_{12} & \bar{h}_{22} \\ h_{22} & l_{22} \end{pmatrix} \begin{pmatrix} f_0\Delta y_0 \\ f_N\Delta y_N \end{pmatrix} \quad (4.13)$$

for all $(H|L)$ in

$$\mathbf{T}_{\mathbf{A}}\mathcal{B}^{\mathbb{C}} = \mathbf{T}_{\mathbf{A}}\mathcal{O}_{1,3}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & h_{12} & 0 & \bar{h}_{22} \\ 0 & h_{22} & 0 & l_{22} \end{pmatrix} : h_{12}, l_{22} \in \mathbb{R}, h_{22} \in \mathbb{C} \right\};$$

(2) when $\mathbf{A} \in \mathcal{O}_{1,4}^{\mathbb{C}}$,

$$d\Lambda|_{\mathbf{A}}(H|L) = (f_0\Delta\bar{y}_0, \bar{y}_N) \begin{pmatrix} h_{12} & \bar{h}_{22} \\ h_{22} & l_{21} \end{pmatrix} \begin{pmatrix} f_0\Delta y_0 \\ y_N \end{pmatrix} \quad (4.14)$$

for all $(H|L)$ in

$$\mathbf{T}_{\mathbf{A}}\mathcal{B}^{\mathbb{C}} = \mathbf{T}_{\mathbf{A}}\mathcal{O}_{1,4}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & h_{12} & \bar{h}_{22} & 0 \\ 0 & h_{22} & l_{21} & 0 \end{pmatrix} : h_{12}, l_{21} \in \mathbb{R}, h_{22} \in \mathbb{C} \right\};$$

(3) when $\mathbf{A} \in \mathcal{O}_{2,3}^{\mathbb{C}}$,

$$d\Lambda|_{\mathbf{A}}(H|L) = (\bar{y}_0, f_N\Delta\bar{y}_N) \begin{pmatrix} h_{11} & \bar{h}_{21} \\ h_{21} & l_{22} \end{pmatrix} \begin{pmatrix} y_0 \\ f_N\Delta y_N \end{pmatrix} \quad (4.15)$$

for all $(H|L)$ in

$$\mathbf{T}_{\mathbf{A}}\mathcal{B}^{\mathbb{C}} = \mathbf{T}_{\mathbf{A}}\mathcal{O}_{2,3}^{\mathbb{C}} = \left\{ \begin{pmatrix} h_{11} & 0 & 0 & \bar{h}_{21} \\ h_{21} & 0 & 0 & l_{22} \end{pmatrix} : h_{11}, l_{22} \in \mathbb{R}, h_{21} \in \mathbb{C} \right\};$$

(4) when $\mathbf{A} \in \mathcal{O}_{2,4}^{\mathbb{C}}$,

$$d\Lambda|_{\mathbf{A}}(H|L) = (\bar{y}_0, \bar{y}_N) \begin{pmatrix} h_{11} & \bar{h}_{21} \\ h_{21} & l_{21} \end{pmatrix} \begin{pmatrix} y_0 \\ y_N \end{pmatrix} \quad (4.16)$$

for all $(H|L)$ in

$$\mathbf{T}_{\mathbf{A}}\mathcal{B}^{\mathbb{C}} = \mathbf{T}_{\mathbf{A}}\mathcal{O}_{2,4}^{\mathbb{C}} = \left\{ \begin{pmatrix} h_{11} & 0 & \bar{h}_{21} & 0 \\ h_{21} & 0 & l_{21} & 0 \end{pmatrix} : h_{11}, l_{21} \in \mathbb{R}, h_{21} \in \mathbb{C} \right\}.$$

Proof. We first show that (4.13) holds. For $\mathbf{A} = [A \mid B] \in \mathcal{O}_{1,3}^{\mathbb{C}}$ given by (2.6), we define

$$\mathbf{B} := [A + H \mid B + L] = \begin{bmatrix} 1 & a_{12} + h_{12} & 0 & \bar{z} + \bar{h}_{22} \\ 0 & z + h_{22} & -1 & b_{22} + l_{22} \end{bmatrix},$$

where $h_{12}, l_{22} \in \mathbb{R}$, $h_{22} \in \mathbb{C}$. Obviously, $\mathbf{B} \rightarrow \mathbf{A}$ as $(h_{12}, h_{22}, l_{22}) \rightarrow 0$. By Remark 4.2, we can choose an eigenfunction $z = z(\cdot, \mathbf{B})$ of $\Lambda(\boldsymbol{\omega}, \mathbf{B})$ such that $z \rightarrow y$ as $\mathbf{B} \rightarrow \mathbf{A}$. From (1.1), we get

$$(\Lambda(\mathbf{B}) - \Lambda(\mathbf{A})) \sum_{n=1}^N w_n y_n \bar{z}_n = [y_0, z_0] - [y_N, z_N],$$

which, together with the boundary conditions,

$$A \begin{pmatrix} y_0 \\ f_0 \Delta y_0 \end{pmatrix} + B \begin{pmatrix} y_N \\ f_N \Delta y_N \end{pmatrix} = 0,$$

and

$$(A + H) \begin{pmatrix} z_0 \\ f_0 \Delta z_0 \end{pmatrix} + (B + L) \begin{pmatrix} z_N \\ f_N \Delta z_N \end{pmatrix} = 0,$$

implies that

$$\begin{aligned} & (\Lambda(\mathbf{B}) - \Lambda(\mathbf{A})) \sum_{n=1}^N w_n y_n \bar{z}_n \\ &= (f_0 \Delta y_0)(f_0 \Delta \bar{z}_0) h_{12} + (f_N \Delta y_N)(f_0 \Delta \bar{z}_0) \bar{h}_{22} \\ & \quad + (f_0 \Delta y_0)(f_N \Delta \bar{z}_N) h_{22} + (f_N \Delta y_N)(f_N \Delta \bar{z}_N) l_{22}. \end{aligned}$$

Further, using the following equalities

$$\partial/\partial z = (1/2)(\partial/\partial z_1 - i\partial/\partial z_2), \quad \partial/\partial \bar{z} = 1/2(\partial/\partial z_1 + i\partial/\partial z_2),$$

where $z = z_1 + iz_2$ with $z_1, z_2 \in \mathbb{R}$, one can easily conclude that (4.13) holds. With similar arguments, one can show that (4.14), (4.15), and (4.16) hold. This proof is complete. \square

Next, we give an important application of Theorem 4.5.

Theorem 4.6. *Assume that (1.1) is in $\Omega_N^{\mathbb{R},+}$. Then, in each of the coordinate systems $\mathcal{O}_{1,3}^{\mathbb{C}}$, $\mathcal{O}_{1,4}^{\mathbb{C}}$, $\mathcal{O}_{2,3}^{\mathbb{C}}$, and $\mathcal{O}_{2,4}^{\mathbb{C}}$ in $\mathcal{B}^{\mathbb{C}}$, every continuous eigenvalue branch is always increasing in the two real axis directions.*

For example, in $\mathcal{O}_{1,3}^{\mathbb{C}}$, every continuous eigenvalue branch is always increasing in the a_{12} -direction and in the b_{22} -direction. Note that the monotonicity in Theorem 4.6 is not necessarily strict (see Example 5.4).

Proof. Let $z \in \mathbb{C}$ and $b_{22} \in \mathbb{R}$. By Lemma 3.7 in [18] we know that

$$C_{z,b_{22}} := \left\{ \begin{bmatrix} 1 & s & 0 & \bar{z} \\ 0 & z & -1 & b_{22} \end{bmatrix} : s \in \mathbb{R} \right\} \cup \left\{ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & b_{22} \end{bmatrix} \right\}$$

is a real-analytic loop. Let Λ be a continuous eigenvalue branch on a subset of $C_{z,b_{22}}$. Note that both $C_{z,b_{22}}$ and the curve

$$\lambda \mapsto [\Phi_N(\lambda) \mid -I], \quad \lambda \in \mathbb{R},$$

are real-analytic. So, either their intersection is discrete in $C_{z,b_{22}}$ or they agree completely.

In the former case, Λ is simple on a dense subset of its domain. Fix an $s_0 \in \mathbb{R}$ and $\delta > 0$ such that

$$\mathbf{A}(s) := \begin{bmatrix} 1 & s & 0 & \bar{z} \\ 0 & z & -1 & b_{22} \end{bmatrix}, \quad s \in (s_0 - \delta, s_0 + \delta),$$

lies in the domain of Λ when $\Lambda(\mathbf{A}(s_0))$ is simple. Assume that y is a normalized eigenfunction for $\Lambda(\mathbf{A}(s_0))$. Then, by (4.13) we get that

$$d\Lambda|_{\mathbf{A}}((H|L)) = |f_0 \Delta y_0|^2 \geq 0.$$

Since Λ is simple in a dense open subset of $(s_0 - \delta, s_0 + \delta)$, a similar argument implies that Λ has a non-negative derivative at each point in the dense open subset. Thus, Λ is increasing in the dense open subset. Assume that $\Lambda(\mathbf{A}(s_1))$ is an eigenvalue of multiplicity 2 with $s_1 \in (s_0 - \delta, s_0 + \delta)$. By the continuity of Λ one has that

$$\lim_{s \rightarrow s_1^-} \Lambda(\mathbf{A}(s)) = \Lambda(\mathbf{A}(s_1)) = \lim_{s \rightarrow s_1^+} \Lambda(\mathbf{A}(s)),$$

which, together with the monotonicity of Λ in the dense open subset, implies that Λ is increasing in a neighborhood of s_1 . Thus, Λ is increasing in $(s_0 - \delta, s_0 + \delta)$.

The latter case may happen for at most one pair $z \in \mathbb{R}$ and $b_{22} \in \mathbb{R}$. The monotonicity of Λ can be deduced from the former case by perturbing z or b_{22} . For example, by perturbing b_{22} , namely by b'_{22} , Λ is increasing on the domain according to discussion in the former case. So, setting that $s_1 < s_2$, we have that

$$\Lambda(\mathbf{A}(s_1, b_{22})) = \lim_{b'_{22} \rightarrow b_{22}} \Lambda(\mathbf{A}(s_1, b'_{22})) \leq \lim_{b'_{22} \rightarrow b_{22}} \Lambda(\mathbf{A}(s_2, b'_{22})) = \Lambda(\mathbf{A}(s_2, b_{22})).$$

One can show the rest of the claims similarly. The proof is complete. \square

4.3. Monotonicity on Sturm-Liouville equations of continuous eigenvalue branches of self-adjoint discrete SLPs

Lemma 4.4. *If u and v are eigenfunctions for eigenvalues of two self-adjoint discrete SLPs $((1/f, q, w), \mathbf{A})$ and $((1/g, r, s), \mathbf{A})$, respectively, with the same BC \mathbf{A} , then*

$$u_0(g_0 \Delta \bar{v}_0) - (f_0 \Delta u_0) \bar{v}_0 = u_N(g_N \Delta \bar{v}_N) - (f_N \Delta u_N) \bar{v}_N. \quad (4.17)$$

Proof. First, consider the coupled self-adjoint BC \mathbf{A} . It follows from Lemma 2.1 that $\mathbf{A} = [e^{i\gamma} K \mid -I]$ for some $\gamma \in [0, \pi)$ and $K \in \text{SL}(2, \mathbb{R})$. So, we have that

$$K^t E K = E,$$

$$\begin{pmatrix} u_N \\ f_N \Delta u_N \end{pmatrix} = e^{i\gamma} K \begin{pmatrix} u_0 \\ f_0 \Delta u_0 \end{pmatrix}, \quad \begin{pmatrix} v_N \\ g_N \Delta v_N \end{pmatrix} = e^{i\gamma} K \begin{pmatrix} v_0 \\ g_0 \Delta v_0 \end{pmatrix}.$$

Thus,

$$\begin{aligned}
& (g_N \Delta \bar{v}_N) u_N - \bar{v}_N (f_N \Delta u_N) = \begin{pmatrix} v_N \\ g_N \Delta v_N \end{pmatrix}^* E \begin{pmatrix} u_N \\ f_N \Delta u_N \end{pmatrix} \\
& = \begin{pmatrix} v_0 \\ g_0 \Delta v_0 \end{pmatrix}^* K^t E K \begin{pmatrix} u_0 \\ f_0 \Delta u_0 \end{pmatrix} = \begin{pmatrix} v_0 \\ g_0 \Delta v_0 \end{pmatrix}^* E \begin{pmatrix} u_0 \\ f_0 \Delta u_0 \end{pmatrix} \\
& = (g_0 \Delta \bar{v}_0) u_0 - \bar{v}_0 (f_0 \Delta u_0).
\end{aligned}$$

Hence, (4.17) holds in the coupled case.

The separated case can be treated similarly. The proof is complete. \square

Theorem 4.7. Fix $\mathbf{A} \in \mathcal{B}^{\mathbb{C}}$. Let λ_* be a simple eigenvalue of a self-adjoint discrete SLP $(\omega, \mathbf{A}) = ((1/f, q, w), \mathbf{A})$, $y \in l[0, N+1]$ a normalized eigenfunction for λ_* , and Λ the continuous simple eigenvalue branch over $\Omega_N^{\mathbb{R},+}$ through λ_* . Then,

$$d\Lambda|_{\omega}(h, k, l) = - \sum_{n=0}^{N-1} |f_n \Delta y_n|^2 h_n + \sum_{n=1}^N |y_n|^2 k_n - \lambda_* \sum_{n=1}^N |y_n|^2 l_n \quad (4.18)$$

for all $(h, k, l) = ((h_0, \dots, h_N), (k_1, \dots, k_N), (l_1, \dots, l_N)) \in \mathbf{T}_{\omega} \Omega_N^{\mathbb{R},+} = \mathbb{R}^{N+1} \times \mathbb{R}^N \times \mathbb{R}^N$.

Proof. Denote $\Lambda = \Lambda(1/f, q, w)$ and $y = y(\cdot, 1/f, q, w)$. By Remark 4.2 we can choose an eigenfunction $z = y(\cdot, 1/f + h, q + k, w + l)$ with respect to $\lambda(1/f + h, q + k, w + l)$ for $(h, k, l) \in \mathbb{R}^{N+1} \times \mathbb{R}^N \times \mathbb{R}^N$ sufficiently small such that $z \rightarrow y$ as $(h, k, l) \rightarrow 0$. For convenience, we set $1/g = 1/f + h$ with $g = \{g_n\}_{n=0}^N$, $\hat{q} = q + k$, $\hat{w} = w + l$. Using (1.1) and Lemma 4.4, we get that

$$\begin{aligned}
& (\Lambda(1/g, \hat{q}, \hat{w}) - \Lambda(1/f, q, w)) \sum_{n=1}^N w_n y_n \bar{z}_n \\
& = \sum_{n=1}^N (\bar{z}_n \nabla(f_n \Delta y_n) - y_n \nabla(g_n \Delta \bar{z}_n)) - \Lambda(1/g, \hat{q}, \hat{w}) \sum_{n=1}^N l_n y_n \bar{z}_n + \sum_{n=1}^N k_n y_n \bar{z}_n \\
& = \sum_{n=1}^N \Delta y_{n-1} (g_{n-1} \Delta \bar{z}_{n-1}) - \sum_{n=1}^N \Delta \bar{z}_{n-1} (f_{n-1} \Delta y_{n-1}) + y_0 (g_0 \Delta \bar{z}_0) \\
& \quad - y_N (g_N \Delta \bar{z}_N) + \bar{z}_N (f_N \Delta y_N) - \bar{z}_0 (f_0 \Delta y_0) - \Lambda(1/g, \hat{q}, \hat{w}) \sum_{n=1}^N l_n y_n \bar{z}_n + \sum_{n=1}^N k_n y_n \bar{z}_n \\
& = \sum_{n=0}^{N-1} (f_n \Delta y_n) (g_n \Delta \bar{z}_n) (1/f_n - 1/g_n) - \Lambda(1/g, \hat{q}, \hat{w}) \sum_{n=1}^N l_n y_n \bar{z}_n + \sum_{n=1}^N k_n y_n \bar{z}_n \\
& = - \sum_{n=0}^{N-1} (f_n \Delta y_n) (g_n \Delta \bar{z}_n) h_n - \Lambda(1/g, \hat{q}, \hat{w}) \sum_{n=1}^N l_n y_n \bar{z}_n + \sum_{n=1}^N k_n y_n \bar{z}_n,
\end{aligned}$$

which yields that (4.18) holds. This completes the proof. \square

Theorem 4.8. *Fix a self-adjoint BC. Then, each continuous eigenvalue branch Λ over $\Omega_N^{\mathbb{R},+}$ is decreasing in every $(1/f_n)$ -direction with $0 \leq n \leq N-1$, independent of f_N , and increasing in every q_n -direction; while the positive parts of Λ are decreasing in every w_n -direction, and the negative parts of Λ are increasing in every w_n -direction.*

Proof. The proof is similar to that of Theorem 4.6 and hence omitted. \square

Remark 4.3. The monotonicity in Theorem 4.8 is not necessarily strict. Please see Examples 5.5-5.7 for illustration.

5. Examples

In this section, we shall give some examples to illustrate some results obtained in Sections 3 and 4.

Consider the modified discrete Fourier equation, i.e., the discrete SLE (1.1) with

$$N = 2, f_0 = f_1 = 1, q_1 = q_2 = 0, w_1 = 1.$$

From (3.1) and (3.2) we deduce that

$$\begin{aligned} \Phi_0(\lambda) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \Phi_1(\lambda) = \begin{pmatrix} 1 & 1 \\ -\lambda & 1 - \lambda \end{pmatrix}, \\ \Phi_2(\lambda) &= \begin{pmatrix} 1 - \lambda & 2 - \lambda \\ -(1 + w_2)\lambda + w_2\lambda^2 & 1 - (1 + 2w_2)\lambda + w_2\lambda^2 \end{pmatrix}. \end{aligned} \quad (5.1)$$

Further, when $w_2 = 1$, we have that

$$\Phi_2(\lambda) = \begin{pmatrix} 1 - \lambda & 2 - \lambda \\ -2\lambda + \lambda^2 & 1 - 3\lambda + \lambda^2 \end{pmatrix}. \quad (5.2)$$

We first give two examples to show that the analytic and geometric multiplicities of an eigenvalue are not necessarily equal for a discrete SLP, which is not self-adjoint.

Example 5.1. Consider the modified discrete Fourier equation with $w_2 = 1$, and let $c \in \mathbb{C}$. Then, by using Lemma 3.3 and (5.2), the characteristic function for the separated BC

$$\mathbf{A}(c) := \begin{bmatrix} c & 2c+1 & 0 & 0 \\ 0 & 0 & c & 1 \end{bmatrix}$$

is

$$\Gamma(\lambda) = (c^2 + 2c + 2)\lambda - (c + 1)\lambda^2.$$

Thus, 0 is an eigenvalue for $\mathbf{A}(-1 \pm i)$ with geometric multiplicity 1 and analytic multiplicity 2. So, the analytic and geometric multiplicities of an eigenvalue are not equal in general. Note that the BCs $\mathbf{A}(-1 \pm i)$ are not self-adjoint.

Example 5.2. Take the discrete SLP consisting of the modified discrete Fourier equation and the separated BC

$$\begin{bmatrix} \cos \alpha & 2 \cos \alpha - \sin \alpha & 0 & 0 \\ 0 & 0 & \cos \alpha & -\sin \alpha \end{bmatrix},$$

where $\alpha \in [0, \pi)$. By using Lemma 3.3 and (5), direct calculations yield that the characteristic function of the problem is

$$\Gamma(\lambda) = (1 - \sin(2\alpha) + w_2 \sin^2 \alpha)\lambda + (\cos \alpha - \sin \alpha)w_2 \sin \alpha \cdot \lambda^2.$$

Thus, if

$$\alpha \in (0, \pi/4) \cup (\pi/4, \pi), \quad w_2 = (\sin(2\alpha) - 1)/\sin^2 \alpha, \quad (5.3)$$

then 0 is an eigenvalue with geometric multiplicity 1 and analytic multiplicity 2. In this case, the weight function w is indefinite since $w_2 < 0$ by (5.3).

The following examples below show us continuous eigenvalue branches in different cases.

Example 5.3. Consider the discrete Fourier equation with $\omega_2 = 1$, and let $\alpha \in \mathbb{R}$. Then, from Lemma 3.3 and (5.2), we see that the characteristic function for the BC $\mathbf{S}_{\alpha, \pi}$ (see (2.5)) is

$$\Gamma(\lambda) = -(1 - \lambda) \sin \alpha - (2 - \lambda) \cos \alpha = -2 \cos \alpha - \sin \alpha + (\cos \alpha + \sin \alpha)\lambda.$$

Thus, the self-adjoint discrete SLP consisting of the discrete Fourier equation and $\mathbf{S}_{3\pi/4, \pi}$ has no eigenvalues; and when $\alpha \in [0, 3\pi/4) \cup (3\pi/4, \pi)$, the only eigenvalue for $\mathbf{S}_{\alpha, \pi}$ is

$$\lambda_1 = (2 \cos \alpha + \sin \alpha)/(\cos \alpha + \sin \alpha). \quad (5.4)$$

So, in this case, there is only one continuous eigenvalue branch over $\{\mathbf{S}_{\alpha, \pi} : \alpha \in [0, 3\pi/4) \cup (3\pi/4, \pi)\}$, i.e., the function given by (5.4) for $\alpha \in [0, 3\pi/4) \cup (3\pi/4, \pi)$. See Figure 5.1.

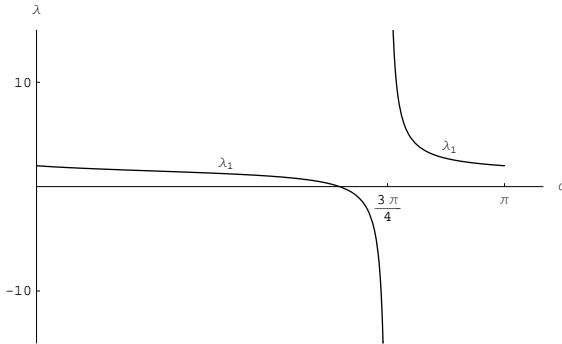


Figure 5.1. Only one continuous eigenvalue branch

Similarly, the characteristic function for the BC $\mathbf{S}_{\alpha, \pi/2}$ is

$$\Gamma(\lambda) = -\cos \alpha + (3 \cos \alpha + 2 \sin \alpha)\lambda - (\cos \alpha + \sin \alpha)\lambda^2.$$

Thus, the only eigenvalue for $\mathbf{S}_{3\pi/4, \pi/2}$ is $\lambda_1 = 1$; and when $\alpha \in [0, 3\pi/4) \cup (3\pi/4, \pi)$, the two eigenvalues for $\mathbf{S}_{\alpha, \pi/2}$ are

$$\lambda_1(\alpha) = \begin{cases} \lambda_-(\alpha) & \text{if } \alpha \in [0, 3\pi/4), \\ \lambda_+(\alpha) & \text{if } \alpha \in (3\pi/4, \pi), \end{cases} \quad \lambda_2(\alpha) = \begin{cases} \lambda_+(\alpha) & \text{if } \alpha \in [0, 3\pi/4), \\ \lambda_-(\alpha) & \text{if } \alpha \in (3\pi/4, \pi), \end{cases}$$

$$\lambda_{\pm}(\alpha) = \frac{3 \cos \alpha + 2 \sin \alpha \pm \sqrt{\cos^2 \alpha + 4 \sin(2\alpha) + 4}}{2(\cos \alpha + \sin \alpha)}.$$

So, in this case, each continuous eigenvalue branch over $\{\mathbf{S}_{\alpha, \pi/2} : \alpha \in [0, \pi)\}$ is locally a part of one of the following functions:

$$\begin{aligned} \Lambda_1(\alpha) &= \lambda_1(\alpha) \text{ for } \alpha \in [0, 3\pi/4), \\ \Lambda_{2,1}(\alpha) &= \begin{cases} \lambda_2(\alpha) & \text{if } \alpha \in [0, 3\pi/4), \\ 1 & \text{if } \alpha = 3\pi/4, \\ \lambda_1(\alpha) & \text{if } \alpha \in (3\pi/4, \pi), \end{cases} \\ \Lambda_2(\alpha) &= \lambda_2(\alpha) \text{ for } \alpha \in (3\pi/4, \pi), \end{aligned}$$

See Figure 5.2. Note that for each $\mathbf{S}_{\alpha, \pi/2}$, there are one or two continuous eigenvalue branches defined on a neighborhood of $\mathbf{S}_{\alpha, \pi/2}$ in $\{\mathbf{S}_{\alpha, \pi/2} : \alpha \in [0, \pi)\}$. This example demonstrates that the index of the eigenvalue in a continuous eigenvalue branch over $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ can change as the problem varies.

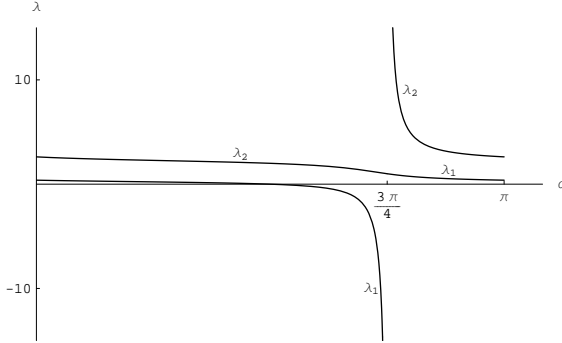


Figure 5.2. One or two continuous eigenvalue branches

Note that Figures 5.1 and 5.2 also agree with the strict monotonicity of continuous eigenvalue branches in the α -direction given in Theorem 4.4.

The next example shows that the monotonicity in Theorem 4.6 is not necessarily strict.

Example 5.4. Consider the discrete Fourier equation with $\omega_2 = 1$. Let $a_{12} > 1$ and $b_{21} \in \mathbb{R}$. Then, the characteristic function for the BC

$$\mathbf{A}(a_{12}, b_{21}) := \begin{bmatrix} 1 & a_{12} & -1 & 0 \\ 0 & -1 & b_{21} & 1 \end{bmatrix} \in \mathcal{O}_{1,4}^{\mathbb{C}}$$

is

$$\Gamma(\lambda) = -(a_{12} - 2)b_{21} + [(a_{12} - 1)b_{21} + 2(a_{12} - 2)]\lambda - (a_{12} - 1)\lambda^2.$$

Thus, the two eigenvalues for $\mathbf{A}(a_{12}, b_{21})$ are

$$\lambda_1(a_{12}, b_{21}) = \frac{(a_{12} - 1)b_{21} + 2(a_{12} - 2) - \delta^{\frac{1}{2}}(a_{12}, b_{21})}{2(a_{12} - 1)}, \quad (5.5)$$

$$\lambda_2(a_{12}, b_{21}) = \frac{(a_{12} - 1)b_{21} + 2(a_{12} - 2) + \delta^{\frac{1}{2}}(a_{12}, b_{21})}{2(a_{12} - 1)}, \quad (5.6)$$

where

$$\delta(a_{12}, b_{21}) = (a_{12} - 1)^2 b_{21}^2 + 4(a_{12} - 2)^2.$$

Let $\Lambda_1(a_{12}, b_{21}) = \lambda_1(a_{12}, b_{21})$ and $\Lambda_2(a_{12}, b_{21}) = \lambda_2(a_{12}, b_{21})$. These are the two continuous eigenvalue branches over

$$\{\mathbf{A}(a_{12}, b_{21}) : a_{12} > 1, b_{21} \in \mathbb{R}\}.$$

Let $R = (1, +\infty) \times \mathbb{R}$. Since $\delta(a_{12}, b_{21}) > 0$ for each $(a_{12}, b_{21}) \in R \setminus \{(2, 0)\}$, then $\lambda_1(a_{12}, b_{21})$ and $\lambda_2(a_{12}, b_{21})$ are two different and simple eigenvalues in this case. By Lemma 4.2 and Theorem 4.2, Λ_1 and Λ_2 are the only two different C^∞ eigenvalue branches on $R \setminus \{(2, 0)\}$.

On the other hand, setting $b_{21} = 0$, from (5.5) and (5.6) we deduce that

$$\lambda_1(a_{12}, 0) = \begin{cases} 0 & \text{if } a_{12} > 2, \\ 2(a_{12} - 2)/(a_{12} - 1) & \text{if } 1 < a_{12} < 2, \end{cases}$$

$$\lambda_2(a_{12}, 0) = \begin{cases} 2(a_{12} - 2)/(a_{12} - 1) & \text{if } a_{12} > 2, \\ 0 & \text{if } 1 < a_{12} < 2. \end{cases}$$

See Figure 5.3. Therefore, the two continuous eigenvalue branches $\lambda_1(a_{12}, 0)$ and $\lambda_2(a_{12}, 0)$ are not differentiable at $a_{12} = 2$. Note that $\lambda = 0$ is the eigenvalue of multiplicity 2 of the problem when $a_{12} = 2$ and $b_{21} = 0$. This demonstrates that the multiplicity assumptions in Theorems 4.1 and 4.2 can not be omitted in general (see Remark 4.1).

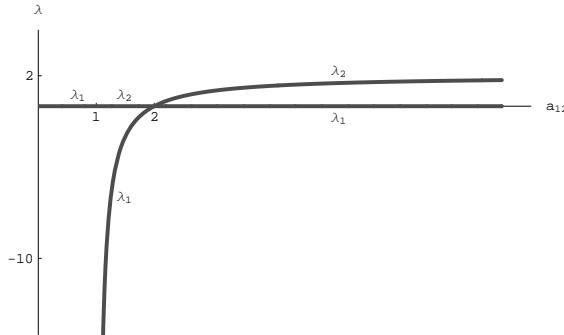


Figure 5.3. One or two continuous eigenvalue branches, with horizontal parts

Figure 5.3 also illustrates the monotonicity of continuous eigenvalue branches in the a_{12} -direction in $\mathcal{O}_{1,4}^C$. Moreover, this example shows that the monotonicity is not strict in general.

Finally, we give three examples to show different continuous eigenvalue branches on different subsets of $\Omega_N^{\mathbb{R},+}$ of the discrete self-adjoint SLP, separately.

Example 5.5. Let $s < 0$. Consider the 1-parameter family of self-adjoint discrete SLPs consisting of the discrete SLEs with

$$f_0 = s, \quad f_1 = 1, \quad f_2 = 1, \quad q_1 = q_2 = 0, \quad w_1 = w_2 = 1, \quad N = 2,$$

and the BC

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \in \mathcal{O}_{1,3}^{\mathbb{C}}. \quad (5.7)$$

Then, by Lemma 3.3, direct calculations deduce that the characteristic function is

$$\Gamma(\lambda) = (-1 + \lambda)(1 + s\lambda)/s.$$

Thus, the two continuous eigenvalue branches are

$$\lambda_1(s) = \begin{cases} -1/s & \text{if } s \leq -1, \\ 1 & \text{if } -1 < s < 0, \end{cases} \quad \lambda_2(s) = \begin{cases} 1 & \text{if } s \leq -1, \\ -1/s & \text{if } -1 < s < 0. \end{cases}$$

See Figure 5.4. Therefore, in general, continuous eigenvalue branches are not differentiable with respect to f_n , and their monotonicity with respect to f_n is not strict.

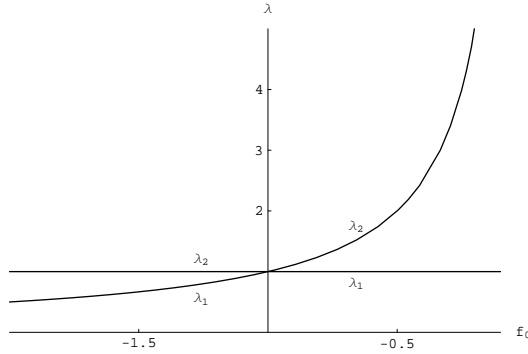


Figure 5.4. Continuous eigenvalue branches are increasing in f_0 -direction

Example 5.6. Let $s \in \mathbb{R}$. Take the 1-parameter family of self-adjoint discrete SLPs consisting of the discrete SLEs with

$$f_0 = -1, \quad f_1 = 1, \quad f_2 = 1, \quad q_1 = s, \quad q_2 = 0, \quad w_1 = w_2 = 1, \quad N = 2,$$

and the BC given in (5.7). Then, the characteristic function is

$$\Gamma(\lambda) = (1 - \lambda)(1 + s - \lambda).$$

Thus, the two continuous eigenvalue branches are

$$\lambda_1(s) = \begin{cases} 1 + s & \text{if } s \leq 0, \\ 1 & \text{if } s > 0, \end{cases} \quad \lambda_2(s) = \begin{cases} 1 & \text{if } s \leq 0, \\ 1 + s & \text{if } s > 0. \end{cases}$$

See Figure 5.5. Therefore, in general, continuous eigenvalue branches are not differentiable with respect to q_n , and their monotonicity with respect to q_n is not strict.

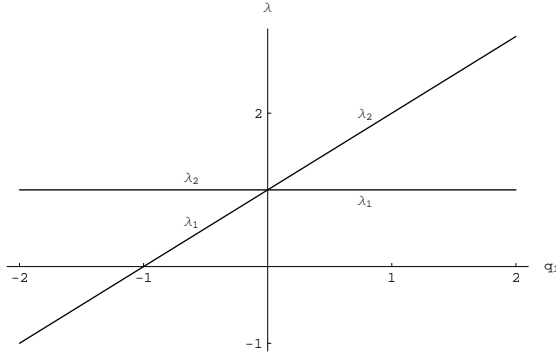


Figure 5.5. Continuous eigenvalue branches are increasing in q_1 -direction

Example 5.7. Let $s > 0$. Consider the 1-parameter family of self-adjoint discrete SLPs consisting of the discrete SLEs with

$$f_0 = -1, \quad f_1 = 1, \quad f_2 = 1, \quad q_1 = q_2 = 0, \quad w_1 = s, \quad w_2 = 1, \quad N = 2,$$

and the BC given in (5.7). Then, the characteristic function is

$$\Gamma(\lambda) = (1 - \lambda)(1 - s\lambda).$$

Thus, the two continuous eigenvalue branches are

$$\lambda_1(s) = \begin{cases} 1 & \text{if } 0 < s \leq 1, \\ 1/s & \text{if } s > 1, \end{cases} \quad \lambda_2(s) = \begin{cases} 1/s & \text{if } 0 < s \leq 1, \\ 1 & \text{if } s > 1. \end{cases}$$

See Figure 5.6. Therefore, in general, continuous eigenvalue branches are not differentiable with respect to w_n , and the monotonicity of their positive parts with respect to w_n is not strict.

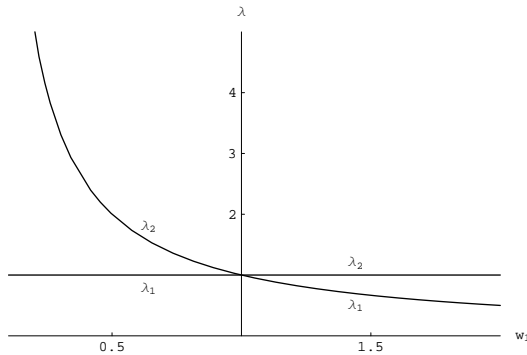


Figure 5.6. Positive parts of continuous eigenvalue branches are decreasing in w_1 -direction

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